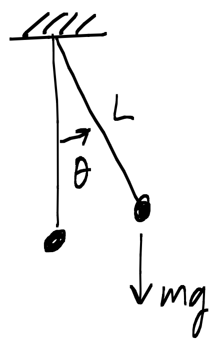
Lesson 3 Linear v.s. non-linear

Linear: $(e^x + x^3)\frac{dy}{dx} + (\cos x - x^{10})\,y = (\ln x + x^{20})$

$$A_3(x)\frac{d^3y}{dx^3} + A_2(x)\frac{d^2y}{dx^2} + A_1(x)\frac{dy}{dx} + A_0(x)\,y = B(x)$$

Non-linear: $\frac{dy}{dx} = y^2 \leftarrow$ square ruins linearity in y.



$F = ma$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta \leftarrow \text{non-linear}$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$\frac{\sin\theta}{\theta} \to 1$ as $\theta \to 0$. For small θ, $\sin\theta \approx \theta$.

$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$ is the "linearized eqn"

EX: $\frac{dy}{dx} = y^2 \leftarrow$ non-linear. Separable!

$$\int \underbrace{\frac{1}{y^2}}_{y^{-2}}\,dy = \int dx$$

$$\frac{1}{-2+1}y^{-2+1} = x + C$$

Book in 1.2 does

$$\underbrace{\frac{1}{y^2}\frac{dy}{dx}}_{\frac{d}{dy}\left(-\frac{1}{y}\right)} = 1$$

Chain rule!

$$-\frac{1}{y} = x + C$$

$$y = \frac{-1}{x+C}$$

$$\frac{d}{dy}\left(-\frac{1}{y}\right)\frac{dy}{dx} = 1$$

$$\frac{d}{dx}\left[-\frac{1}{y}\right] = 1$$

So $-\frac{1}{y} = \int 1\, dx = x + C$

Find sol^n with $y(1) = 3$.

Plug in $x=1$, $y=3$:

$$3 = \frac{-1}{1+C}$$

$$-\frac{1}{3} = 1 + C$$

$$\boxed{C = -\frac{4}{3}}$$

$$y = \frac{-1}{x - \frac{4}{3}}$$

Ouch! Blows up at $x = \frac{4}{3}$

This is a danger for non-linear eqns.

__EX:__ $\frac{dy}{dx} = ky$. Sol^n $y = Ce^{kx}$ defined for all x

linear

__EX:__ First order linear ODE with constant coeff

$$A\frac{dy}{dx} + By = C, \quad A, B, C \text{ constants}$$

Divide out by A. Get

$$\frac{dy}{dx} + by = c \qquad \text{where } b = \frac{B}{A},\ c = \frac{C}{A}.$$

Separable? $\frac{dy}{dx} = c - by$

$$\int \frac{dy}{\underbrace{c-by}_{u}} = \int dx \qquad \text{Yes!}$$

$du = -b\,dy$

so $dy = -\frac{1}{b}du$

$$\int \frac{-\frac{1}{b}du}{u} = x + K$$

$$-\frac{1}{b}\ln|u| = x + K$$

$u = c - by$

$$\ln|c-by| = -bx - bK$$

$e \qquad e$

$$|c-by| = e^{-bx}e^{-bK}$$

$$c - by = \underbrace{\pm e^{-bK}}_{\tilde{K}} e^{-bx}$$

$$y = \frac{c}{b} \underbrace{\mp \frac{\tilde{K}}{b}}_{+K} e^{-bx}$$

$$\boxed{y = \frac{c}{b} + Ke^{-bx}}$$

Ugh! Separable method was messy for this linear eqn.

EX: $x^2 \frac{dy}{dx} + 2xy = e^{2x}$

$$\underbrace{x^2 \frac{dy}{dx} + 2xy}_{\frac{d}{dx}[x^2 y]}$$

Luck!

So $x^2 y = \int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$

$$y = \frac{e^{2x}}{2x^2} + \frac{C}{x^2}$$

What? This sol^n is bad at $x=0$. Why?

To see what goes bad at $x=0$, put ODE in $\frac{dy}{dx} = f(x,y)$ form:

$$x^2 \frac{dy}{dx} + 2xy = e^{2x}$$

$$\frac{dy}{dx} = \frac{e^{2x} - 2xy}{x^2}$$

Aha! D.Field has bad behavior when $x=0$.

Method for solving linear first order ODE

$$A(x)\frac{dy}{dx} + B(x)y = C(x)$$

Step 1: Put eqn in standard form: Divide by $A(x)$:

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{where } \begin{cases} P = \frac{B}{A} \\ Q = \frac{C}{A} \end{cases}$$

Forget about A, B, C.

Step 2: Look for a $u(x)$ such that when you multiply eqn by $u(x)$:

$$\underbrace{u\frac{dy}{dx} + uPy}_{} = uQ$$

Want this to be $\frac{d}{dx}[uy] = u\frac{dy}{dx} + u'y$

Aha! Need $u' = uP$

$$\boxed{u' = uP}$$

$$\underbrace{\frac{u'}{u}}_{\frac{d}{dx}\ln|u|} = P$$

So $e^{\ln|u|} = e^{\int P\,dx}$

No $+C$ here. Just want one stupid u that works and I want the cleanest simplest one. Take $C=0$.

$$|u| = e^{\int P\,dx}$$

$$u = \pm e^{\int P\,dx}$$

Take the $+$ one

$$\boxed{u = e^{\int P\,dx}}$$ works! Use it.