Lesson 4 Linear first order ODE

Hwk 1 due Wed. (Stapled as one.)

Linear 1-st order ODE $A(x)\frac{dy}{dx} + B(x)y = C(x)$

Step 1: Put in standard form. (Divide by $A(x)$)

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)} \qquad \left\{ \begin{array}{l} P = \frac{B}{A} \\ Q = \frac{C}{A} \end{array} \right.$$

Step 2: Cook up an "integrating factor" u:

Multiply Standard form eqn by u:

$$\underbrace{u\frac{dy}{dx} + uPy}_{\text{want}} = uQ$$

$$= \frac{d}{dx}[uy] = u\frac{dy}{dx} + \frac{du}{dx}y$$

wish $\frac{du}{dx} = uP$

$$\underbrace{x^2\frac{dy}{dx} + (2x)y}_{\frac{d}{dx}[x^2y]} = x^3$$

so

$$x^2y = \int x^3\,dx = \frac{1}{4}x^4 + C$$

No sweat! Want $\underbrace{\frac{1}{u}\frac{du}{dx}}_{\frac{d}{dx}\ln|u|} = P$

So $\ln|u| = \int P\,dx$

Take $C=0$ in $+C$ part, Just want the nicest u that works.

$$|u| = e^{\int P\,dx}$$

$$u = \pm e^{\int P\,dx}$$

Take $\boxed{u = e^{\int P\,dx}}$ ← Memorize: 1) Standard form to get P.
2) formula for u

EX: $\frac{1}{3}\frac{dy}{dx} + y = \frac{1}{3}e^{2x}$

Step 1: Multiply by $\frac{1}{(1/3)} = 3$ to put in S.F.:

$$\boxed{\frac{dy}{dx} + \underset{\uparrow\, P(x)=3}{3}y = e^{2x}}$$

Step 2: Get $u = e^{\int P(x)\,dx} = e^{\int 3\,dx} = e^{3x}$

Step 3: Multiply S.F. eqn by u:

$$\underbrace{e^{3x}\frac{dy}{dx} + (3e^{3x})y}_{\frac{d}{dx}[\underset{\uparrow\, u}{e^{3x}}\, y]} = e^{3x}\cdot e^{2x} = e^{5x}$$

↑ Don't forget to multiply by u here

$$\frac{d}{dx}[e^{3x}y] = e^{5x}$$

So $\quad e^{3x} y = \int e^{5x}\,dx + C$

$= \frac{1}{5} e^{5x} + C$

want all possible y's

and $\quad y = \frac{1}{5}\frac{e^{5x}}{e^{3x}} + \frac{C}{e^{3x}}$

$$\boxed{y = \frac{1}{5} e^{2x} + C e^{-3x}}$$

Gen^l sol^n

Infinitely many sol^n s! Need an initial condition to pin down a unique sol^n.

EX: Find sol^n with $y(0) = 1$.

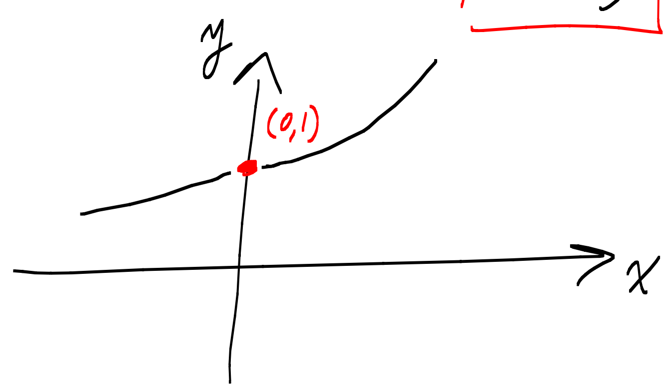
Plug $x = 0$, $y = 1$ into gen^l sol^n: $\quad 1 = \frac{1}{5} e^{2\cdot 0} + C e^{-3\cdot 0}$

$1 = \frac{1}{5} + C$

$\boxed{C = \frac{4}{5}}$

$$\boxed{y = \frac{1}{5} e^{2x} + \frac{4}{5} e^{-3x}}$$

$y(0) = 1$ ✓



Don't recommend this formula:

$$y = e^{-\int P(x)\,dx} \int e^{\int P(x)\,dx} Q\,dx + C e^{-\int P(x)\,dx}$$

It just summarizes the steps above.

EX: $$\frac{dy}{dx} + \underline{\underline{e^{-x^2}}} y = 1$$

$P(x) =$

$$u = e^{\int e^{-x^2} dx}$$ No formula

Nicest antiderivative of e^{-x^2} is

$$\text{erf}(x) = \int_0^x e^{-t^2} dt$$

EX: $$\frac{dy}{dt} \underline{\underline{-(\tan t)}} y = 8 \sin^3 t$$

Luck. In S.F.

$P(t) = -\tan t$

$$u = e^{\int P(t)\,dt} = e^{\int -\tan t\,dt} = e^{-\ln|\sec t|}$$

$$= e^{\ln|\sec t|^{-1}} = |\underbrace{\sec t}_{\frac{1}{\cos t}}|^{-1} = |\cos t|$$

So $u = \pm \cos t$. Take the + one.

$$\boxed{u = \cos t}$$

u. (S.F.) : $$\cos t \frac{dy}{dt} - \cos t \tan t\, y = 8 \cos t \sin^3 t$$

$$\frac{d}{dt}\left[\cos t \; y\right] = 8 \cos t \sin^3 t$$

So $(\cos t)\, y = \int 8 \cos t \sin^3 t \, dt$

$v = \sin t$

$dv = \cos t \, dt$

$$= 8\int v^3 \, dv = 8 \cdot \left[\tfrac{1}{4} v^4\right] + C$$

$$= 2 \sin^4 t + C$$

Finally, $y = 2 \dfrac{\sin^4 t}{\cos t} + \dfrac{C}{\cos t}$

$$\boxed{y = 2 \sin^3 t \tan t + C \sec t}$$

<u>Advice</u>: Linear eqns are separable, but the integrals and formulas are nicer in the linear method of today.

<u>EX</u>: $2 \dfrac{dy}{dx} + 3y = 5$ ← Separable

$$\int \frac{2\, dy}{5 - 3y} = \int dx$$

S.F. $\boxed{\dfrac{dy}{dx} + \dfrac{3}{2} y = \dfrac{5}{2}}$

$$u = e^{\int \frac{3}{2}dx} = e^{\frac{3}{2}x}$$

$$\underbrace{\boxed{e^{\frac{3}{2}x}\frac{dy}{dx} + \frac{3}{2}e^{\frac{3}{2}x}y}}_{\frac{d}{dx}\left[e^{\frac{3}{2}x}y\right]} = \frac{5}{2}e^{\frac{3}{2}x} \leftarrow u\cdot(\text{S.F.})$$

So $$e^{\frac{3}{2}x}y = \int \frac{5}{2}e^{\frac{3}{2}x}\,dx$$

$$= \frac{5}{2}\cdot\frac{2}{3}e^{\frac{3}{2}x} + C$$

and $$\boxed{y = \frac{5}{3} + Ce^{-\frac{3}{2}x}}$$

<u>Techniques of integration</u>: $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$

$$\int \frac{1}{y^2+y}\,dy = ?$$

Partial fractions

$$\frac{1}{y^2+y} = \frac{1}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1}$$

Mult. by denom $y(y+1)$: $1 = A(y+1) + By$

$$0 \cdot y + 1 = \underbrace{(A+B)}_{\text{must } = 0} y + \underbrace{A}_{\text{must } = 1}$$

$$A + B = 0 \leftarrow \underline{\underline{B = -A = -1}}$$

$$\boxed{A = 1}$$

$$\int \frac{1}{y^2+y}\,dy = \int \left(\frac{1}{y} - \frac{1}{y+1}\right) dy$$

$$= \ln|y| - \ln|y+1| + C$$

$$= \ln\left|\frac{y}{y+1}\right| + C$$