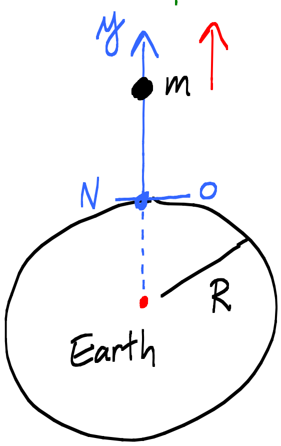
Lesson 5 Applications of separable equations

Read Chap 2
HWK 2 on home page
due next Wed



Shoot a cannonball into space

$$F = ma = m\frac{dv}{dt}$$

$$\frac{-GMm}{(y+R)^2} = m\frac{dv}{dy}\frac{dy}{dt}$$ ← Chain rule

Aha! $v = \frac{dy}{dt}$

$$\frac{-GMm}{(y+R)^2} = mv\frac{dv}{dy}$$ ← Separable!

$$\int -\frac{GMm}{(y+R)^2}\,dy = \int mv\,dv$$

$$\underbrace{\frac{GMm}{(y+R)}}_{\text{Gravitational potential energy}} = \underbrace{\frac{1}{2}mv^2}_{\text{Kinetic energy}} + C$$ ← Stop! Find C now!

C comes from initial conditions. At $t=0$, we shoot from surface of earth. So $y=0$. Initial v: $v = v_0$.

Get

$$\frac{GMm}{(0+R)} = \frac{1}{2}mv_0^2 + C$$

$$\boxed{C = \frac{GMm}{R} - \frac{1}{2}mv_0^2}$$

Solve for $v = \sqrt{\frac{2GM}{(y+R)} - \frac{2GM}{R} + v_0^2}$

How high does cannonball go? At max y_{max}, $\frac{dy}{dt} = 0$

$v = 0$.

Set $v=0$: Get $\frac{2GM}{(y_{max}+R)} - \frac{2GM}{R} + v_0^2 = 0$

$$\frac{2GM}{(y_{max}+R)} = \underbrace{\frac{2GM}{R} - v_0^2}$$

Hmmm. Get y_{max} if $v_0^2 < \frac{2GM}{R}$.

Wow! If $v_0 = \sqrt{\frac{2GM}{R}}$, the ball never stops to turn around!

Escape velocity!

If $v_0 \geq \sqrt{\frac{2GM}{R}}$, the ball "escapes".

Rocket scientists: $v = \sqrt{\;\ell\ell\ell\;}$ ← separable!

$v = \frac{dy}{dt}$

$$\int \frac{dy}{\sqrt{\frac{2GM}{(y+R)} - \frac{2GM}{R} + v_0^2}} = \int dt = t + C_2$$

Hairy! Get an implicit formula for sol^n $y(t)$

But, $\int$ cleans up nicely if v_0 = escape velocity. (Next time)

Homogeneous first order ODE:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \longleftarrow \text{homogeneous fcn}$$

Method: Change variables: Try $v = \frac{y}{x}$ ← fcn of x, ← x is x!

v = new fcn of x

Step 1: Change $\frac{dy}{dx}$ to something involving v:

$$v = \frac{y}{x}$$

So $y = xv$

$$\frac{dy}{dx} = \frac{dx}{dx}\cdot v + x\cdot\frac{dv}{dx} = v + x\frac{dv}{dx}$$

$$\boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}}$$

Step 2: Plug into ODE. (Eliminate y)

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

$$\boxed{v + x\frac{dv}{dx} = F(v)} \longleftarrow \text{Separable}$$

Step 3: Solve it

$$\int \frac{dv}{F(v)-v} = \int \frac{1}{x}\,dx = \ln|x| + C$$

Advice. Get C now.

Step 4: Solve for v. $v =$ fcn of x

Step 5: $\left(\frac{y}{x}\right) =$ fcn of x

Solve for y. Done!

EX: $$\frac{dy}{dx} = e^{y/x} + \frac{y}{x}$$

$$v + x\frac{dv}{dx} = e^{v} + v$$

$$x\frac{dv}{dx} = e^{v}$$

$$\int \frac{dv}{e^{v}} = \int \frac{dx}{x} = \ln|x| + C$$

$$\underbrace{\int e^{-v}\,dv}_{-e^{-v}} = \ln|x| + C$$

$$e^{-v} = -\ln|x| - C$$

So $\quad -v = \ln e^{-v} = \ln\left(-\ln|x| - c\right)$

$$v = -\ln\left(-\ln|x| - c\right)$$

$v = \frac{y}{x}$

$$\boxed{y = -x\ln\left(-\ln|x| - c\right)}$$

In the homework: ODE in form

$$\left(x^2 + 3xy + y^2\right)dx + \left(-x^2\right)dy = 0$$

Notation for $\quad (\quad) + (\quad)\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2} = 1 + 3\frac{y}{x} + \left(\frac{y}{x}\right)^2$$

Homogeneous!

EX: $\quad \frac{dy}{dx} = \ln y - \ln x = \ln\frac{y}{x} \quad \leftarrow$ homogeneous

EX: $\quad \frac{dy}{dx} = \frac{x+y}{x-y}\cdot\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \frac{1 + \left(\frac{y}{x}\right)}{1 - \left(\frac{y}{x}\right)} \quad \leftarrow$ homog

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$

$$x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v-v(1-v)}{1-v} = \frac{1+v^2}{1-v}$$

$$\underbrace{\int \frac{1-v}{1+v^2}\,dv}_{} = \int \frac{dx}{x} = \ln|x| + C$$

$$\int \frac{1}{1+v^2} - \frac{2v}{2(1+v^2)}\,dv$$

$$\text{ArcTan}\,v - \tfrac{1}{2}\ln(1+v^2) = \ln|x| + C$$

$$\text{ArcTan}\,\frac{y}{x} - \ln\sqrt{1+\left(\frac{y}{x}\right)^2} = \ln|x| + C$$ Ugh!

Defines sol^n $y(x)$ impicitely!