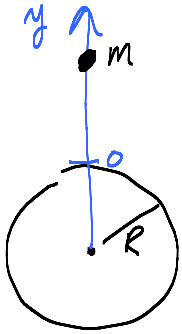
Lesson 6 Applications

HWK 2 on home page. Due Wed.



$F = ma$

$$\frac{-GMm}{(y+R)^2} = m\frac{dv}{dt} = m\frac{dv}{dy}\underbrace{\frac{dy}{dt}}_{v}$$ ← Separable

Got $\quad \dfrac{v^2}{2} = \dfrac{GM}{R+y} + C$

$v = v_0$ when $y = 0$

Get $C = \dfrac{v_0^2}{2} - \dfrac{GM}{R}$

$$(*) \qquad v = \sqrt{\frac{2GM}{R+y} + \underbrace{\left(v_0^2 - \frac{2GM}{R}\right)}}$$

$v = \dfrac{dy}{dt}$

< 0, ball turns around and comes back

≥ 0, ball never turns around!

Escape velocity: $v_e = \sqrt{\dfrac{2GM}{R}}$

Note: $\dfrac{GMm}{R^2} = mg$

Prob: Find $y(t)$ when $v_0 = \sqrt{\dfrac{2GM}{R}}$.

$(*)$ becomes: $\quad v = \sqrt{\dfrac{2GM}{y+R}}$ ← Separable!

$= \dfrac{dy}{dt}$

$$\int \sqrt{y+R}\, dy = \int \sqrt{2GM}\, dt = \sqrt{2GM}\, t + K$$

Find K now!

$$\frac{1}{\frac{1}{2}+1}(y+R)^{\frac{1}{2}+1} =$$

Initial condition: $y=0$ when $t=0$.

$$\frac{2}{3}R^{3/2} = 0 + K$$

So $$\frac{2}{3}(y+R)^{3/2} = \sqrt{2GM}\,t + \frac{2}{3}R^{3/2}$$

Solve for $$\boxed{y = \left[\frac{3}{2}\sqrt{2GM}\,t + R^{3/2}\right]^{2/3} - R}$$

Now we see $\lim\limits_{t\to\infty} y(t) = \infty$.

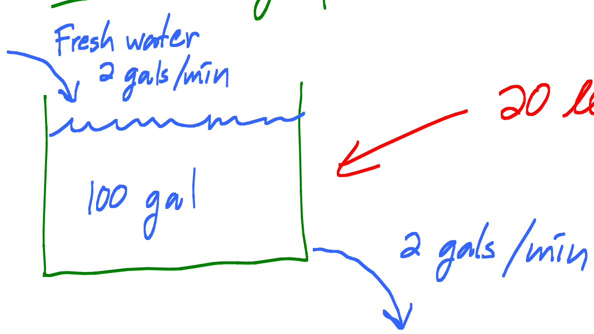
and $$v = \sqrt{\frac{2GM}{(y+R)}} \longrightarrow 0 \text{ as } t\to\infty$$

(because $y(t)\to\infty$).

What if $v_0 > v_e$? Can show $y(t)\to\infty$ as $t\to\infty$ too.

$$v = \sqrt{\frac{2GM}{(y+R)} + \left(v_0^2 - \frac{2GM}{R}\right)} \longrightarrow \underbrace{\sqrt{v_0^2 - \frac{2GM}{R}}}_{\text{like a "terminal velocity"}} \text{ as } t\to\infty$$

Modeling: EX: Mixing problems



20 lbs salt in tank at time $t=0$

$S(t) =$ lbs of salt in tank at time t. Given $\underline{S(0) = 20}$

$$\frac{dS}{dt} = (\text{Rate in}) - (\text{Rate out})$$

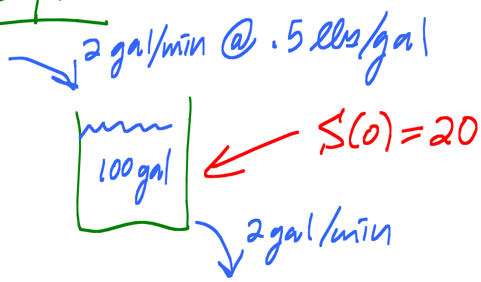
$$= \left(2\,\frac{\text{gal}}{\text{min}} \cdot 0\,\frac{\text{lbs}}{\text{gal}}\right) - \left(2\,\frac{\text{gals}}{\text{min}}\right) \cdot \left(\frac{S(t)}{100\text{ gal}}\right)$$

$$\boxed{\frac{dS}{dt} = -\frac{1}{50} S}$$

$$S(t) = S_0 e^{-\frac{1}{50}t} = 20 e^{-t/50}$$

See that $S(t) \to 0$ as $t \to \infty$, but never reaches 0!

Same prob but hose in salty water with concentration $.5$ lbs/gal.



$$\frac{dS}{dt} = (\text{Rate in}) - (\text{Rate out})$$

$$= (2\text{ gal/min})(.5\text{ lbs/gal}) - (2\text{ gals/min})\left(\frac{S(t)}{100\text{ gal}}\right)$$

$$\boxed{\frac{dS}{dt} + \underbrace{\frac{1}{50}}_{P(t) = \frac{1}{50}} S = 1}$$

Linear 1st order ODE with constant coeff in <u>Standard Form</u>.

Int factor: $u = e^{\int P\,dt} = e^{\int \frac{1}{50}dt} = e^{t/50}$

$$\underbrace{e^{t/50}\left(\frac{dS}{dt} + \frac{1}{50}S\right)} = 1 \cdot e^{t/50}$$

$$\frac{d}{dt}\left[e^{t/50}S\right] = e^{t/50}$$

So $$e^{t/50}S = \int e^{t/50}dt + C$$

$$= 50e^{t/50} + C$$

and $$S(t) = 50 + Ce^{-t/50}$$

Get C from initial cond: $S(0) = 20$

$$20 = 50 + Ce^{0}$$

$$\boxed{C = -30}$$

Sol^{n}: $$\boxed{S(t) = 50 - 30e^{-t/50}}$$

See $S(t) \to 50$ as $t \to \infty$.

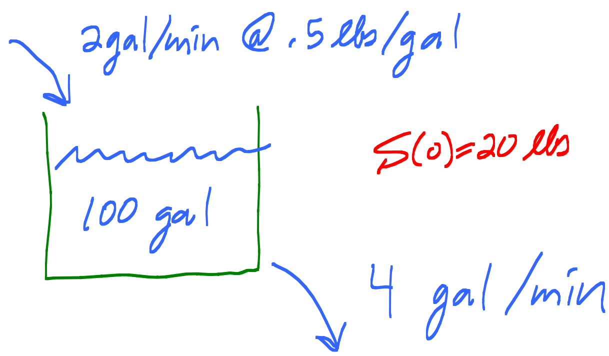
Hmmm. Expected: $(.5\text{ lbs/gal})(100\text{ gal}) = 50\text{ lbs.}$

What is the limiting concentration of salt in the tank?

$$\text{Concent.} = C(t) = \frac{\text{lbs salt}}{\text{vol water}} = \frac{S(t)}{100}$$

$$C(t) = \frac{1}{2} - \frac{3}{100} e^{-t/50} \longrightarrow \frac{1}{2} \frac{\text{lbs}}{\text{gal}} \text{ as } t \to \infty,$$

Prob:



$S(0) = 20$ lbs

Hmmm. Volume $V(t)$ is changing.

$$\frac{dV}{dt} = (\text{Rate in}) - (\text{Rate out})$$
$$= 2 - 4 = -2$$

So $V(t) = \int -2\,dt = -2t + C$

$V(0) = C = 100$ gal.

$$\boxed{V(t) = 100 - 2t}$$

$$\frac{dS}{dt} = (\text{Rate in}) - (\text{Rate out})$$

$$= (2)\cdot(.5) - \left(4\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{S(t)}{\underbrace{V(t)}_{100-2t}}\right)$$

$$\boxed{\frac{dS}{dt} + \underbrace{\frac{4}{100-2t}}_{P(t)} S = 1}$$

Linear 1st order ODE

Int factor $\quad u = e^{\int \frac{2}{50-t}\,dt}$

$$= e^{-2\ln|50-t|} = e^{\ln|50-t|^{-2}}$$

$$= |50-t|^{-2} = \frac{1}{(50-t)^2}$$

What is concentration at instant tank empties?