Lesson 7 Existence & Uniqueness, Linear vs Non-linear

HWK 3 due Wed

Exam 1 in class Monday, Sept. 16 covering first order ODE's (Chap. 1 and 2)

Best case results are for linear eqns:

$$A(x)\frac{dy}{dx} + B(x)\,y = C(x)$$

Divide by $A(x)$ to isolate $\frac{dy}{dx}$ (think $\frac{dy}{dx} = f(x,y)$. D. field)

$$\boxed{\frac{dy}{dx} + P(x)\,y = Q(x)}$$

Standard form. See that zeroes of $A(x)$ could be bad news for the dir. field!

Initial value prob: $y(x_0) = y_0$

Step 1: Integrating factor:

$$u(x) = e^{\int P(x)dx} = e^{\int_{x_0}^{x} P(t)dt}$$

Fund Thm Calculus.

Note: If $P(x)$ is continuous, then $\int_{x_0}^{x} P(t)\,dt$ is continuously differentiable ($\mathcal{C}^1$-smooth).

Consequently, $u(x)$ is $\mathcal{C}^1$-smooth.

$u(x)$ is never zero (because $e^x \neq 0, \forall x$)

$$\left(u(x_0) = e^{\int_{x_0}^{x_0}} = e^0 = 1\right)$$

Step 2: $\underbrace{u\left[y' + Py\right]}_{[uy]'} = uQ$

So $\quad uy = \int uQ\,dx + C$

$$= \int_{x_0}^{x} u(t)\,Q(t)\,dt \quad + C$$

Step 3: Plug in $x = x_0, y = y_0$ to find C:

$$\underbrace{u(x_0)}_{=1}\, y_0 = \underbrace{\int_{x_0}^{x_0} \sim\, dt}_{0} \quad + \quad C$$

So $\boxed{C = y_0.}$

Last step

$$\boxed{y = \frac{1}{u(x)} \int_{x_0}^{x} u(t)\,Q(t)\,dt \quad + \frac{y_0}{u(x)}}$$

Thm: (E & U for first order linear ODE)

$(*)$ $\boxed{\frac{dy}{dx} + P(x)\,y = Q(x), \quad y(x_0) = y_0}$

If $P(x)$ and $Q(x)$ are continuous on (a,b) where $x_0 \in (a,b)$, then there exists a sol^n $y(x)$ to $(*)$ on all of (a,b) that is continuously diff'ble and it is unique.

EX: $x^2 \frac{dy}{dx} + 2xy = 4x^3, \quad y(1) = 3$

Stand. Form: $\frac{dy}{dx} + \frac{2}{x} y = 4x$

boom! at $x=0$

E & U Thm says $y(x)$ is good on $(0,\infty)$ and is unique.

$$u(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{\ln|x|^2} = |x|^2 = x^2$$

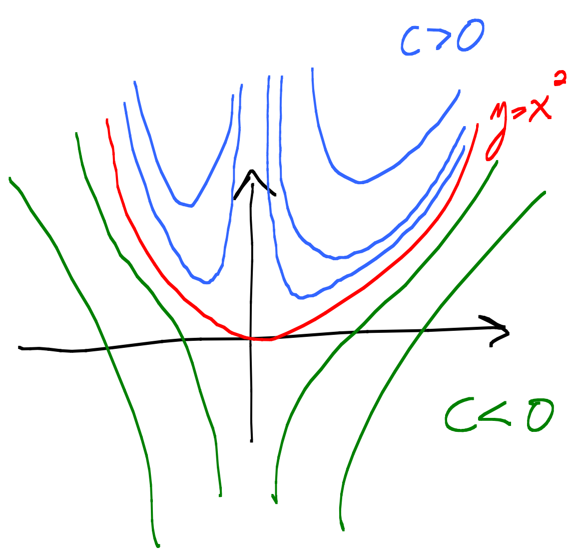
$$\underbrace{x^2\left(y' + \frac{2}{x}y\right)}_{[x^2 y]'} = 4x \cdot x^2 = 4x^3$$

$$x^2 y = \int 4x^3\,dx + C$$

$$= x^4 + C$$

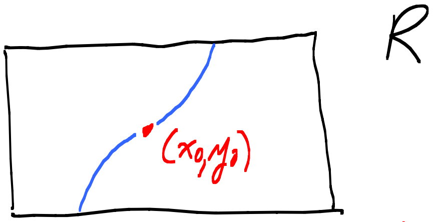
$$y = x^2 + \frac{C}{x^2}$$

if $C \neq 0$, y blows up at $x=0$.



Whoa! If $C=0$, solⁿ is fine at $x=0$.

Non-linear E & U Thm : $\dfrac{dy}{dx} = f(x,y)\,,\quad y(x_0) = y_0 \quad (*)$



$M = \max\limits_{R} |f(x,y)|$

Suppose 1) $f(x,y)$ is continuous on R,

2) $\dfrac{\partial f}{\partial y}$ exists and is continuous on R.

Then a unique sol^n to $(*)$ exists. It is good (continuously diff'ble) as long as it stays in R.

So, all we can say is that there is an $h > 0$ such $y(x)$ exists and is good on $[x_0 - h, x_0 + h]$.

EX: $\dfrac{dy}{dx} = y^2 \leftarrow f(x,y) = y^2 \leftarrow$ contin. ✓

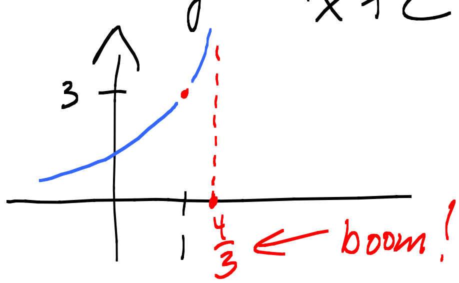
$y(1) = 3 \qquad \dfrac{\partial f}{\partial y} = 2y \leftarrow$ contin. ✓

$$\int \frac{dy}{y^2} = \int dx$$

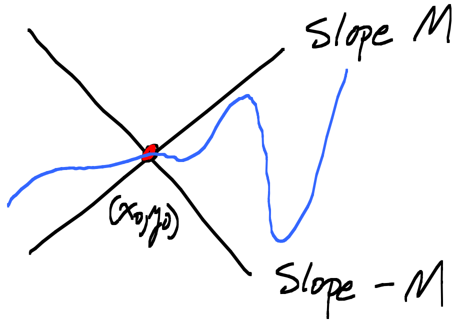
$$-\frac{1}{y} = x + C \leftarrow \begin{matrix} x=1 \\ y=3 \end{matrix} \qquad -\frac{1}{3} = 1 + C \quad \boxed{C = -\frac{4}{3}}$$

$$y = \frac{-1}{x+C} \qquad\qquad y = \frac{-1}{x + \left(-\frac{4}{3}\right)} \leftarrow \text{Ouch!}$$

Game over at $x = \frac{4}{3}$

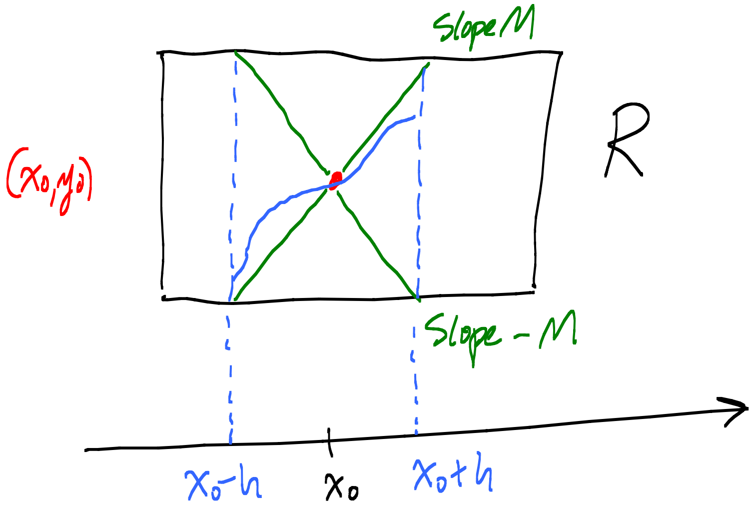


How big an h for $[x_0-h,\ x_0+h]$ can we guarantee?



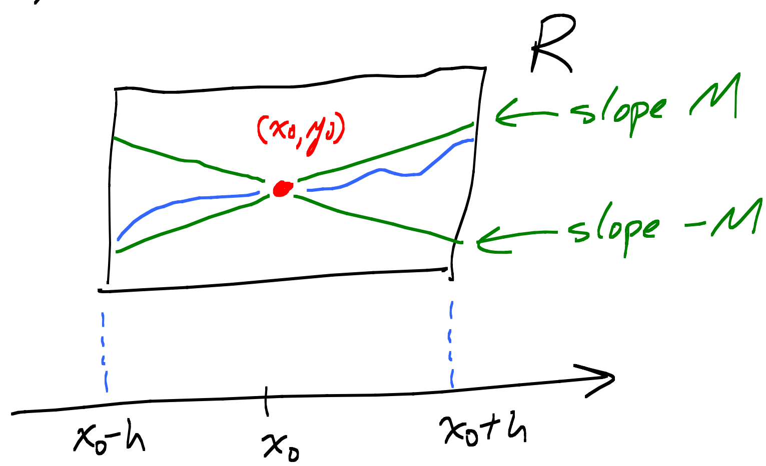
MA 161 : MVT

$$\left|\frac{y(x)-y(x_0)}{x-x_0}\right| = |y'(c)| \le M$$



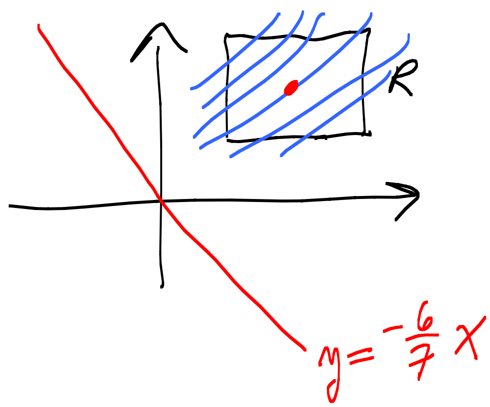
Aha! $y(x)$ is stuck in wedge.

Another possible picture:



EX: $\quad (6x+7y)\dfrac{dy}{dx} - (x-y) = 0$

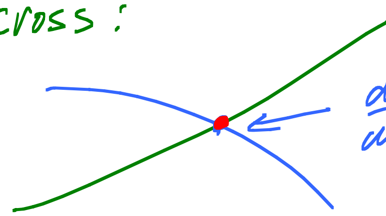
$$\frac{dy}{dx} = \frac{x-y}{6x+7y} \leftarrow f(x,y) \text{ continuous if } 6x+7y \neq 0$$



$$\frac{\partial f}{\partial y} = \cdots = \frac{-7x}{(6x+7y)^2}$$

also continuous away from $y=-\frac{6}{7}x$.

Remarks: Sol^{n}s never cross:



$\frac{dy}{dx}$ = two slopes?

They don't osculate either