Lesson 8 Autonomous first order ODE's, population dynamics

HWK 3 due Wed
Exam 1 on Mon Sept. 16 in class

Autonomous: $\frac{dy}{dt} = f(y)$ ← R.H.S. does not depend on t

EX: $\frac{dy}{dt} = ry$ ← population growth r = rate, const. $\quad y = y(0)e^{rt}$

EX 2: $\frac{dy}{dt} = \underbrace{\left[r\left(1-\frac{y}{K}\right)\right]}_{\text{rate starts out as } r\text{, but shrinks as pop. grows}} y$ ← Logistic eqn

(*) $\frac{dy}{dt} - ry = -\frac{r}{K}y^2$ ($n=2$) ← Bernoulli Eqn

$$\boxed{V = y^{1-2} = y^{-1}}$$

So $\frac{dV}{dt} = \frac{d}{dt}(y^{-1}) = -1\cdot y^{-2}\frac{dy}{dt} = -\frac{1}{y^2}\cdot\frac{dy}{dt}$

Aha! Multiply (*) by $-\frac{1}{y^2}$

$\frac{dy}{dt} + P(t)y = Q(t)y^n$

Change of vars

$V = y^{1-n}$

New ODE is linear in V!

$$\underbrace{-\frac{1}{y^2}\frac{dy}{dt}}_{} + r\underbrace{\frac{1}{y^2}\cdot y}_{\frac{1}{y}=V} = \frac{r}{K}$$

$$\boxed{\frac{dv}{dt} + rv = \frac{r}{K}}$$

Linear! $\quad u = e^{\int r\,dt} = e^{rt}$

$$\underbrace{e^{rt}(\text{L.H.S.})}_{\frac{d}{dt}\left[e^{rt}v\right]} = \frac{r}{K}e^{rt}$$

So $e^{rt} v = \int \frac{r}{K} e^{rt}\, dt = \frac{1}{K} e^{rt} + C$ ← get C:

$v(0) = \frac{1}{y(0)}$

$1 \cdot \frac{1}{y_0} = \frac{1}{K} \cdot 1 + C$

$$\boxed{C = \frac{1}{y_0} - \frac{1}{K}}$$

$$v = \frac{1}{K} + C e^{-rt}$$

$v = \frac{1}{y}$

$$y = \frac{1}{\frac{1}{K} + \left(\frac{1}{y_0} - \frac{1}{K}\right) e^{-rt}} \cdot \frac{y_0 K}{y_0 K}$$

$$\boxed{y = \frac{y_0 K}{y_0 + (K - y_0) e^{-rt}}} \longrightarrow K \text{ as } t \to \infty$$

Qualitative analysis:

Equillibrium sol^{n}s:

$$\underbrace{\frac{dy}{dt}} = \underbrace{f(y)} \leftarrow \text{Suppose } z \text{ is a zero of } f(y)$$

$y \equiv z$ makes RHS = 0

$y \equiv z$ makes LHS = 0 too!

$y \equiv z$, z a zero of $f(y)$ is an equillibrium sol^n.

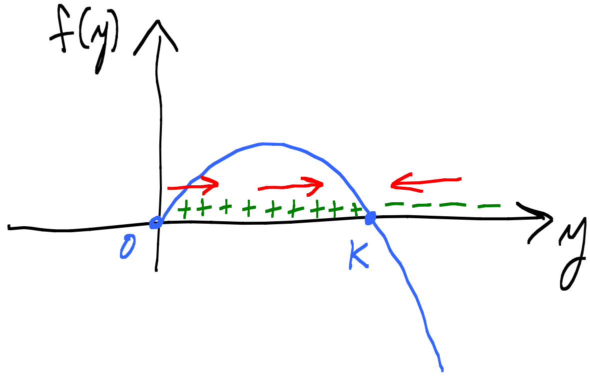
Logistics eqn: $\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right) y$ ← zeroes $y = 0, K$

$y \equiv 0$

$y \equiv K$

equillibrium sol^{n}s

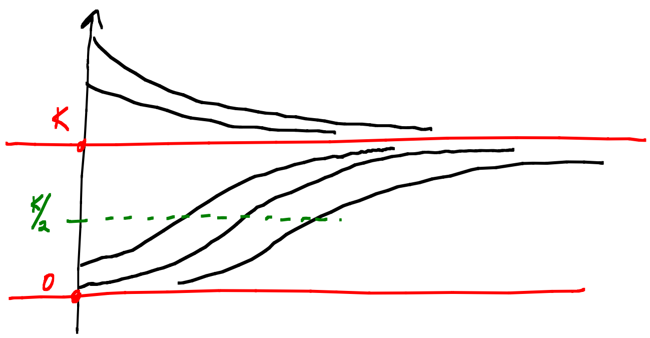
Graph $f(y)$



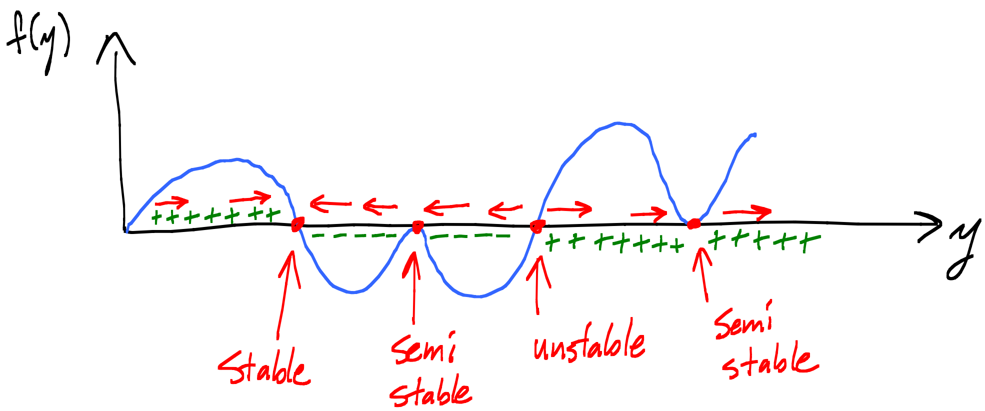
$$\frac{dy}{dt} = f(y)$$

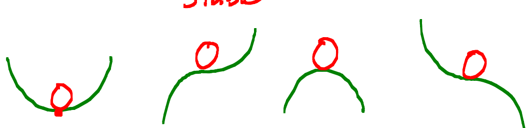
See that $y \equiv K$ is a <u>stable equillibrium</u>.

[Add or substract a few rabbits. $y \to$ back to K.]



<u>General case</u>: $\frac{dy}{dt} = f(y)$





Pin pong ball on a hill

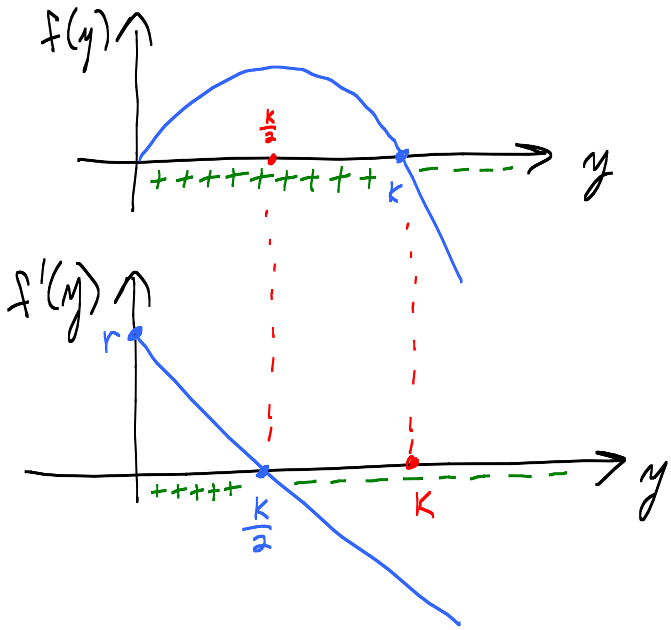
<u>Logistic eqn</u>: <u>Prob</u>: Where are the inflection points?

$$\frac{dy}{dt} = f(y)$$

Idea: Use the chain rule: $\frac{d^2y}{dt^2} = \frac{d}{dt} f(y) = f'(y) \underbrace{\frac{dy}{dt}}_{= f(y)}$

Aha! $$\frac{d^2y}{dt^2} = f'(y)\cdot f(y)$$

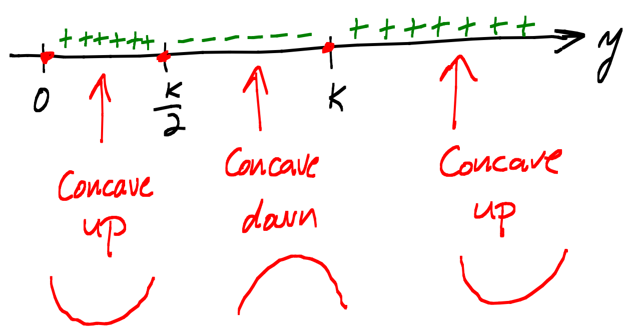
Logistic eqn



$$f(y) = ry - \frac{r}{K}y^2$$

$$f'(y) = r - \frac{2r}{K}y$$

Sign of $\underbrace{f'(y)f(y)}_{\frac{d^2y}{dt^2}}$:



EX: $$\frac{dy}{dt} = -ry\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)$$ ← Carrier pidgeons