Lesson 9 Exact equations

HWK 3 due Wed 9/16
Exam 1 in class next Monday
covering up to today's material 2.6

Problem: Find an ODE solved by family of curves

$$x^2y + y^3 = C$$

← defines fcn y of x implicitly. Write $y(x)$.

Ex: $x^2+y^2=C$

$y = \pm\sqrt{c-x^2}$

Take $\frac{d}{dx}$ of eqn and use implicit diff'tion:

$$\frac{d}{dx}(x^2y) + \frac{d}{dx}(y^3) = \frac{d}{dx}(C)$$

$$\left(2x\cdot y + x^2\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$$

$$\boxed{2xy + (x^2+3y^2)\frac{dy}{dx} = 0}$$

← first order ODE!

Want to go other way!

Prob: $\boxed{M(x,y) + N(x,y)\frac{dy}{dx} = 0}$

Hmmm: Chain rule: $\frac{d}{dx}\varphi(u,v) = \frac{\partial\varphi}{\partial u}\frac{du}{dx} + \frac{\partial\varphi}{\partial v}\frac{dv}{dx}$

$$\frac{d}{dx}\varphi(x,y) = \frac{\partial\varphi}{\partial x}\underbrace{\frac{dx}{dx}}_{1} + \frac{\partial\varphi}{\partial y}\frac{dy}{dx}$$

← y = fcn of x

Above $\varphi(x,y) = C$

$$\boxed{\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{dy}{dx} = 0}$$

Aha! Could we cook up φ so that $\frac{\partial\varphi}{\partial x} = M$, $\frac{\partial\varphi}{\partial y} = N$?

Famous problem: Given $M(x,y) + N(x,y)\frac{dy}{dx} = 0$,

can we find $\varphi(x,y)$ such

$$\begin{cases} \frac{\partial\varphi}{\partial x} = M & (A) \\ \frac{\partial\varphi}{\partial y} = N & (B) \end{cases}$$

If yes, then $\varphi(x,y) = C$ gives sol^{ns} implicitly!

Familiar problem: Given a force field $\vec{F} = M\hat{i} + N\hat{j}$, cook up a potential function φ such that $\vec{F} = \nabla\varphi$.

In $\mathbb{R}^3$: $\vec{F} = \nabla\varphi \iff \vec{F}$ is conservative $\iff$ Curl $\vec{F} = 0$.

In $\mathbb{R}^2$: " " $\iff \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$

How to remember condition $\boxed{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0}$:

If $\nabla\varphi = M\hat{i} + N\hat{j}$, then $\begin{cases} \frac{\partial\varphi}{\partial x} = M & (A) \\ \frac{\partial\varphi}{\partial y} = N & (B) \end{cases}$

So $\frac{\partial}{\partial y}(A) = \frac{\partial^2\varphi}{\partial y\partial x} = \frac{\partial M}{\partial y}$

$\frac{\partial}{\partial x}(B) = \frac{\partial^2\varphi}{\partial x\partial y} = \frac{\partial N}{\partial x}$

Aha! If is φ is C^2-smooth, then MA 261 says $\frac{\partial^2\varphi}{\partial x\partial y} = \frac{\partial^2\varphi}{\partial y\partial x}$

We see that if φ solves (A) and (B), then $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$.

MA 261 showed reverse is true.

Notation and how to remember: $M + N\frac{dy}{dx} = 0$

Multiply by dx: $\underline{\underline{M}}\,dx + \underline{\underline{N}}\,dy = 0$ differential 1-form

$$d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy$$ total differential of φ

Def^n: $M dx + N dy$ is called exact if it is equal to $d\varphi$ for some φ.

How to remember condition $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

$$\frac{\partial}{\partial x}\underbrace{\left[\text{Thing by } dy\right]}_{N} = \frac{\partial}{\partial y}\underbrace{\left[\text{Thing by } dx\right]}_{M}$$

EX: $$\underbrace{(y^2 + \cos x)}_{M} + \underbrace{(2xy + \sin y)}_{N}\frac{dy}{dx} = 0$$

$$\frac{\partial N}{\partial x} = 2y + 0 \overset{?}{=} \frac{\partial M}{\partial y} = 2y + 0 \quad \checkmark \text{ yes!}$$

Can find φ such that

$$\begin{cases} \frac{\partial \varphi}{\partial x} = M = y^2 + \cos x & \text{(A)} \\ \frac{\partial \varphi}{\partial y} = N = 2xy + \sin y & \text{(B)} \end{cases}$$

Step 1: Use (A) :

$$\varphi = \int y^2 + \cos x \, dx$$

"Partial integration" Treat y like a constant in $\int$

$$\boxed{\varphi(x,y) = y^2 \cdot x + \sin x + \underbrace{C(y)}_{\text{an arbitrary fcn of } y!}}$$

((A) is used up.)

Step 2: Use (B) in a totally different way!

$$\frac{\partial \varphi}{\partial y} \overset{\text{want}}{=} 2xy + \sin y \qquad (B)$$

$$\frac{\partial}{\partial y}\left(xy^2 + \sin x + C(y) \right) \overset{\text{want}}{=} 2xy + \sin y$$

$$2xy + 0 + C'(y) = 2xy + \sin y$$

Miracle! All x's cancel! (because of Curl=0 condition!)

$$C'(y) = \sin y$$

$$\text{So} \quad C(y) = \int \sin y \, dy = -\cos y$$

skip $+k$ here. Just want one $\varphi(x,y)$

Step 3: Got $C(y)$! Get φ from red box

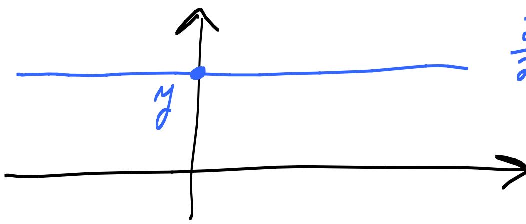
$$\varphi(x,y) = xy^2 + \sin x + \underbrace{(-\cos y)}_{C(y)} = K$$

gives sol^{n}s.

Step 4: Solve for y if possible. (Use initial conditions first.)

Prob: Find all $\Psi(x,y)$ such $\frac{\partial \Psi}{\partial x} = 6xy$.

Lemma: If $\frac{\partial f}{\partial x} \equiv 0$, then f is a fcn of y alone.



$\frac{\partial f}{\partial x} \equiv 0 \implies f = $ const along horiz lines. call it $C(y)$.

Conversely, if $f = C(y)$, then $\frac{\partial f}{\partial x} \equiv 0$.

Sol^n: Want $\frac{\partial \Psi}{\partial x} = 6xy$ Let $f = \int 6xy \, dx = 3x^2 y$ take one

Notice that $\frac{\partial f}{\partial x} = 6xy$ ✓

Aha! $\frac{\partial \Psi}{\partial x} - \frac{\partial f}{\partial x} = 6xy - 6xy \equiv 0$

$$\frac{\partial}{\partial x}\Big(\underbrace{\Psi - 3x^2 y}\Big) \equiv 0$$

must be a fcn $C(y)$ of y. ✓

All possible Ψ's : $3x^2 y + C(y)$

EX: $$\underbrace{(ye^{xy} + 2x)}_{M} + \underbrace{(xe^{xy} - y)}_{N}\frac{dy}{dx} = 0$$

$$\frac{\partial N}{\partial x} = \left(1 \cdot e^{xy} + x \cdot e^{xy} \cdot \underbrace{\frac{\partial}{\partial x}(xy)}_{y}\right) - 0$$

$$\frac{\partial M}{\partial y} = \left(1 \cdot e^{xy} + y \cdot e^{xy} \underbrace{\frac{\partial}{\partial y}(xy)}_{x}\right) + 0$$

= ✓

$$\begin{cases} \dfrac{\partial \varphi}{\partial x} = y e^{xy} + 2x & (A) \\[2ex] \dfrac{\partial \varphi}{\partial y} = x e^{xy} - y & (B) \end{cases}$$

(A) $\varphi = \int y e^{xy} + 2x \, dx = e^{xy} + x^2 + C(y)$

(B) $\dfrac{\partial}{\partial y}\left(e^{xy} + x^2 + C(y) \right) = x e^{xy} - y$ want

$$C'(y) = -y$$

$$C(y) = -\frac{y^2}{2}$$

$$\varphi(x,y) = e^{xy} + x^2 - \frac{1}{2} y^2 = K$$