Lesson 10 Euler's method

Review on Friday, Exam in class Monday

Chap 1, 2 ← 2.1 → 2.6

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \leftarrow \text{shorthand notation}$$

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy$$

Get φ if $\frac{\partial}{\partial x}\left[\begin{matrix}\text{thing}\\ \text{by } dy\end{matrix}\right] = \frac{\partial}{\partial y}\left[\begin{matrix}\text{thing}\\ \text{by } dx\end{matrix}\right]$

$$N_x \overset{?}{=} M_y \quad \leftarrow \text{If yes, eqn is "exact"}$$

What if not exact?

<u>Big idea</u>: Try multiplying $M\,dx + N\,dy = 0$ by an "integrating factor" $u(x,y)$ to <u>make it</u> exact.

$$u\,(M\,dx + N\,dy) = 0$$

$$\underbrace{(uM)}_{\mathcal{M}}dx + \underbrace{(uN)}_{\mathcal{N}}dy = 0$$

Want $\frac{\partial}{\partial x}[uN] \overset{?}{=} \frac{\partial}{\partial y}[uM] \quad (*)$

$$u_x N + u N_x = u_y M + u M_y \quad \leftarrow \text{Ouch! a PDE}$$

<u>Special case</u>: Look for an integrating factor that is a fcn of x alone: $u = u(x)$

Then $(*)$ becomes $u'N + uN_x = 0\cdot M + uM_y$

$$\boxed{\frac{u'}{u} = \frac{M_y - N_x}{N}}$$

Aha! Works if RHS is a fcn of x alone.

Solve separable eqn for u. Then use it.

EX: $\underbrace{(2x-y^2)}_{M} + \underbrace{(xy)}_{N}\frac{dy}{dx} = 0$

$$\frac{u'}{u} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{\overbrace{(-2y)-(y)}^{=-3y \neq 0}}{xy} = \frac{-3}{x}$$

Not exact

fcn of x alone

$$\frac{d}{dx}\operatorname{Ln}|u| = -\frac{3}{x}$$

$$\operatorname{Ln}|u| = \int -\frac{3}{x}\,dx = -3\operatorname{Ln}|x| = \operatorname{Ln}|x|^{-3}$$

$e \qquad e$

$$|u| = |x|^{-3}$$

$$u = \pm\frac{1}{x^3}$$

take $u = \frac{1}{x^3}$

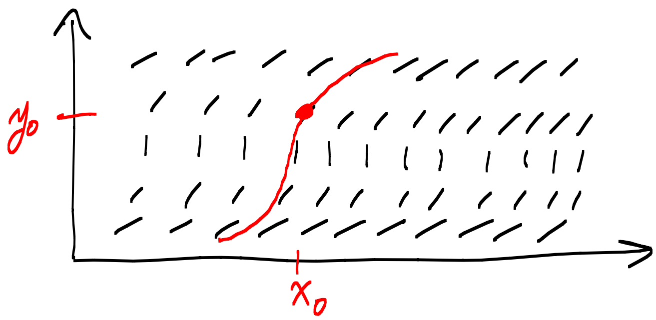
Multiply by $\frac{1}{x^3}$:

$$\frac{2x-y^2}{x^3} + \underbrace{\frac{xy}{x^3}}_{\frac{y}{x^2}}\frac{dy}{dx} = 0$$

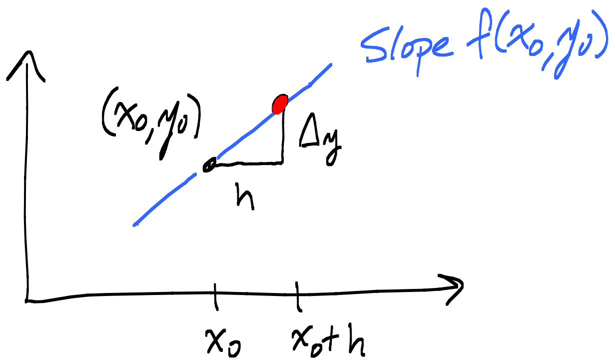
is exact! Solve it.

2.7 Euler's method

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$



Big idea: follow field for a short $\Delta x = h$ at (x_0, y_0)



$$f(x_0,y_0) = \frac{\Delta y}{h}$$

So $\Delta y = h\, f(x_0,y_0)$

Red dot: $\Big(\underbrace{x_0+h}_{x_1},\ \underbrace{y_0 + h\, f(x_0,y_0)}_{y_1}\Big)$

Start again at (x_1, y_1). Continue. Repeat.

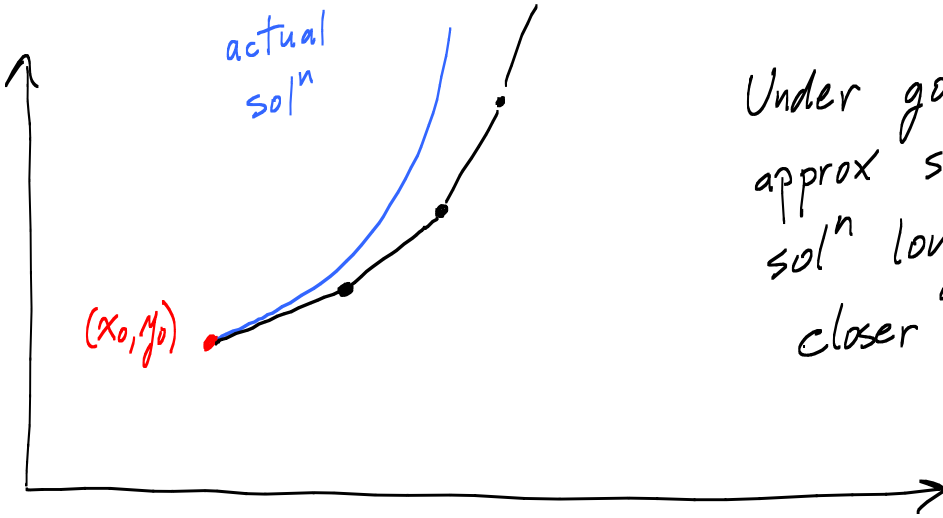
Given x_0, y_0.

$x_1 = x_0 + h, \quad y_1 = y_0 + h\, f(x_0,y_0)$

$x_2 = x_0 + 2h, \quad y_2 = y_1 + h\, f(x_1,y_1)$

$\vdots$

$x_{n+1} = x_n + h, \quad \boxed{y_{n+1} = y_n + h\, f(x_n,y_n)}$ Euler's method



Under good conditions, approx sol^n hugs actual sol^n longer and longer closer and closer.

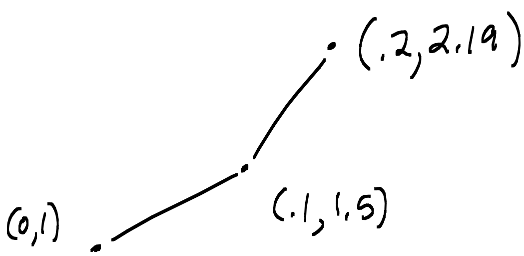
EX: $\dfrac{dy}{dx} = \underbrace{1 - x + 4y}_{f(x,y)}$ $\quad y(\underset{x_0=0}{0}) = \underset{y_0=1}{1}$

Use a step size of $h = .1$

$$y_1 = y_0 + h\,f(x_0,y_0) = 1 + (.1)\big(1 - 0 + 4\cdot 1\big) = 1.5$$

$$y_2 = y_1 + h\,f(x_1,y_1) = (1.5) + (.1)\big(1 - (.1) + 4(1.5)\big) = 2.19$$

$$\vdots$$



etc.

Spreadsheet

n	x_n	y_n	$y_n + h f(x_n, y_n)$
0	0	1	1.5
1	.1	1.5	2.19
2	.2	2.19	3.146
3	.3	3.146	4.4744

MAPLE:

```
x(0) := 0. ;
y(0) := 1. ;
N := 10 ; h := .1 ;

for n from 0 to N-1 do
    x(n+1) := x(n) + h
    y(n+1) := y(n) + h*(1-x(n)+4*y(n))
end do ;
```

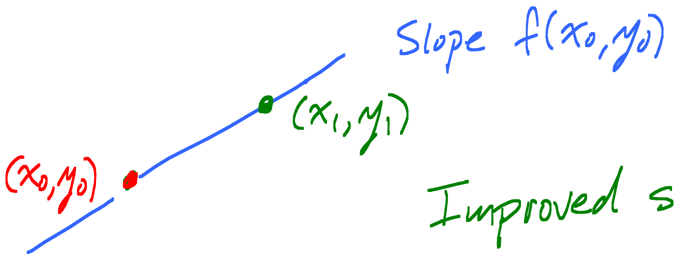
Easy case: $\frac{dy}{dx} = f(x)$, $y(x_0) = y_0$

Euler method yields $y_n = y_0 + \sum_{n=0}^{N-1} f(x_n)\, h$

$\rightarrow y_0 + \int_{x_0}^{x} f(t)\, dt$ ✓

Riemann sum
"Rectangle method"

Improved Euler method



Improved slope guess:

$$\frac{f(x_0,y_0) + f(x_1,y_1)}{2}$$ ← ave

$$x_{n+1} = x_n + h$$

$$\tilde{y}_{n+1} = y_n + h f(x_n, y_n)$$ ← Euler

$$y_{n+1} = y_n + h \cdot \left[\frac{f(x_n,y_n) + f(x_{n+1}, \tilde{y}_{n+1})}{2} \right]$$ ← Improved Euler

Easy case: $\frac{dy}{dx} = f(x)$

Improved Euler method: $y \to y_0 + \underbrace{\int_{x_0}^{x} f(t)\,dt}_{\text{Trapezoid rule!}}$

Hmmm. What corresponds to Simpson's rule?

Answer: Runge-Kutta method.