# Lesson 11 2nd Order linear ODE

HWK 5 due Wed (on homepage)

Easiest 2nd order ODE:
$$\frac{d^2y}{dx^2} = 3x^2$$
$$\frac{dy}{dx} = x^3 + C_1$$

Gen<sup></sup>

Gen$^{l}$ Sol$^{n}$ $\longrightarrow$
$$y = \underbrace{\tfrac{1}{4}x^4}_{\text{A "particular" sol}^n} + \underbrace{C_1 x + C_2}_{\text{Gen}^l\text{ sol}^n\text{ to "homogeneous eq}^n\text{" } y''=0}$$

2nd order <u>linear</u> ODE:
$$A(x)\,\underline{\underline{\frac{d^2y}{dx^2}}} + B(x)\,\underline{\underline{\frac{dy}{dx}}} + C(x)\,\underline{\underline{y}} = \underbrace{D(x)}_{\text{If } D(x)\equiv 0, \text{ then eq}^n \text{ is homogeneous}}$$

**Standard Form**: Divide by $A(x)$:

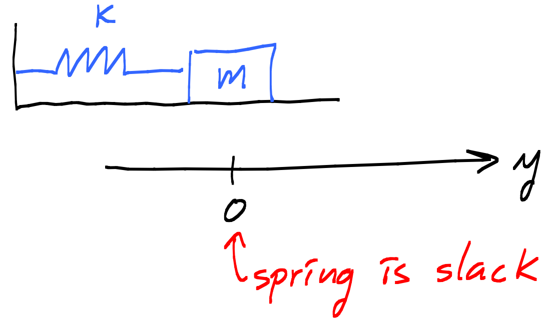
$$(*) \qquad \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = R(x) \qquad \left(P = \frac{B}{A} \ldots\right.$$

**E & U Thm**: The initial value problem: $(*)$ plus
$$\begin{cases} y(x_0) = y_0 \\ y'(x_0) = y_0' \end{cases}$$
has ($\uparrow$ E exists) a unique ($\uparrow$ U) sol$^n$ valid

on largest open interval containing $x_0$ where $P(x), Q(x), R(x)$ are <u>all</u> continuous.

EX: $y'' + 3y' + 2y = 0$

$m=1, \quad c=3, \quad k=2$




net force $\rightarrow F = ma$

$$\underbrace{-ky}_{\text{Hooke's Law}} \quad \underbrace{-c\frac{dy}{dx}}_{\text{friction force}} = m\frac{d^2y}{dx^2}$$

Idea: Try $y = e^{rx}$.

$y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2e^{rx}$

Plug it in and make it work:

$$y'' + 3y' + 2y = 0$$

$$(r^2e^{rx}) + 3(re^{rx}) + 2(e^{rx}) \overset{\text{want}}{=} 0$$

$$[\underbrace{r^2 + 3r + 2}_{\text{Aha! need } = 0.}]\underbrace{e^{rx}}_{\text{never zero}} \equiv 0 \leftarrow \text{zero function}$$

Characteristic polynomial : $\boxed{r^2 + 3r + 2 = 0}$

$(r+2)(r+1) = 0$

Roots : $r = -1, -2$

Get two sol'ns : $y_1 = e^{-x}, \quad y_2 = e^{-2x}$

Hmmm. Is $y = c_1e^{-x} + c_2e^{-2x}$ the gen' sol'n ?

Important property of <u>linear</u> <u>homogeneous</u> eqns :

The superposition principle : If $y_1$ and $y_2$ solve the eqn, so does $c_1 y_1 + c_2 y_2$.

<u>Why</u>: Write $L[y] = y'' + Py' + Qy$

$$L[c_1 y_1 + c_2 y_2] = (c_1 y_1 + c_2 y_2)'' + P(c_1 y_1 + c_2 y_2)' + \dots$$
$$= c_1 y_1'' + c_2 y_2'' + c_1 P y_1' + c_2 P y_2' + \dots$$
$$= c_1 \underbrace{L[y_1]}_{=0} + c_2 \underbrace{L[y_2]}_{=0}$$

Aha! If $y_1, y_2$ solve prob, so does $c_1 y_1 + c_2 y_2$.

<u>Word</u>: $L$ is a <u>linear operator</u>

Now have lots of sol$^n$s : $c_1 e^{-x} + c_2 e^{-2x}$

<u>Big question</u>: Are they <u>all</u> sol$^n$s ?

Equivalent question: (because of E & U).

Can we solve any and all I.V.P. with these ?

Hmmm.

$$y = c_1 e^{-x} + c_2 e^{-2x}$$
$$y' = -c_1 e^{-x} - 2c_2 e^{-2x}$$

$$\begin{cases} y(0) = c_1 + c_2 = A \\ y'(0) = -c_1 - 2c_2 = B \end{cases}$$

want

I.V.P. $\begin{cases} y(0) = A \\ y'(0) = B \end{cases}$

$$\begin{cases} c_1 + c_2 = A \\ -c_1 - 2c_2 = B \end{cases}$$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

Aha! $\text{Det} = 1\cdot(-2) - 1\cdot(-1) = -1 \neq 0$

Think: Cramer's rule:

$$c_1 = \frac{\det\begin{bmatrix} A & 1 \\ B & -2 \end{bmatrix}}{\det\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}} \qquad c_2 = \frac{\det\begin{bmatrix} 1 & A \\ -1 & B \end{bmatrix}}{\det\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}}$$

← need det $\neq 0$ to get uniq sol$^n$ no matter what $A, B$ are

Good news: Can solve any IVP with $y = c_1 e^{-x} + c_2 e^{-2x}$, and so it is the gen$^l$ sol$^n$!

<u>EX</u>: $y'' + 3y' + 2y = 0 \quad \begin{cases} y(0) = 7 \\ y'(0) = 8 \end{cases}$

$$\begin{aligned} c_1 + c_2 &= 7 \\ -c_1 - 2c_2 &= 8 \\ \hline -c_2 &= 15 \end{aligned}$$

$\boxed{c_2 = -15}$

$c_1 = 7 - c_2 = 7 + 15 = 22$

$$\boxed{y = 22e^{-x} - 15e^{-2x}}$$

all sol$^n$s $\to 0$ as $x \to \infty$!

<u>Def$^n$</u>: The Wronskian of $y_1$ and $y_2$

$$W[y_1, y_2](x) = W(x) = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

Relevance: I.V.P. $\begin{cases} y = c_1 y_1 + c_2 y_2 \\ y' = c_1 y_1' + c_2 y_2' \end{cases}$

want

At $x_0$ : $\begin{cases} y(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0) = A \\ y'(x_0) = c_1 y_1'(x_0) + c_2 y_2'(x_0) = B \end{cases}$

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

Aha! Need $W(x_0) \neq 0$ to say I can solve any and all I.V.P.'s. If $W(x_0) \neq 0$, then $c_1 y_1 + c_2 y_2$ is the <u>gen^l sol^n</u>.

Hmmm. $W[y_1, y_2] = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_2 y_1'$

Quotient rule! $\left(\dfrac{y_1}{y_2}\right)' = -\dfrac{W[y_1, y_2]}{y_2^2}$

whoa! If $W \equiv 0$, then $\dfrac{y_1}{y_2} = \text{const.}$

$y_1 = (\text{const})\, y_2 \leftarrow$ linearly dependent

<u>General 2nd order linear homogeous ODE with constant coeff</u>:

$$a y'' + b y' + c y = 0$$

Euler: Try $y = e^{rx}$.

$$\underbrace{\left[ar^2 + br + c\right]}_{=0} e^{rx} \equiv 0$$

Get roots $r_1, r_2$.

Get sol$^n$s $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.

Oh no! What if $r_1 = r_2$?

What if $r = a \pm bi$ complex roots?