Lesson 12 2^{nd} order linear homogeneous ODE with constant coeff

HWK 5 due Wed

EX: $L[y] = y'' + 3y' + 2y = 0 \leftarrow$ homogeneous

Last time: Tried $y = e^{rx}$. Got

$$L[e^{rx}] = (\underbrace{r^2 + 3r + 2}_{\text{need } =0 \text{ Characteristic Eqn}})e^{rx}$$

Got e^{-x}, e^{-2x}

two sol^{n}s

Big fact: $L[y]$ is a linear operator.

$$\begin{cases} L[y_1] = 0 \\ L[y_2] = 0 \end{cases} \quad \text{two sol}^n\text{s}$$

$$L[\underbrace{c_1 y_1 + c_2 y_2}_{\text{sol}^n}] = c_1 \underbrace{L[y_1]}_{0} + c_2 \underbrace{L[y_2]}_{0} = 0$$

Wronskian: If y_1 & y_2 solve prob, and

$$W[y_1, y_2](x_0) = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0)$$

is not zero, then we can cook up sol^n $c_1 y_1 + c_2 y_2$ satisfying any and all I.V.P.'s $\begin{cases} y(x_0) = A \\ y'(x_0) = B \end{cases}$.

Then $c_1 y_1 + c_2 y_2$ is the gen^l sol^n.

EX: $$W[e^{r_1 x}, e^{r_2 x}] = \det \begin{bmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{bmatrix} = r_2 e^{(r_1+r_2)x} - r_1 e^{(r_1+r_2)x}$$

$$= \underbrace{(r_2 - r_1) e^{(r_1+r_2)x}}_{\text{never zero if } r_1 \neq r_2.}$$

Good news: $ay'' + by' + cy = 0$

Char. Eqn: $ar^2 + br + c = 0$

Roots r_1, r_2 real and <u>unequal</u>. $(b^2 - 4ac > 0)$

Then $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ is the gen^l sol^n.

<u>EX</u>: $y'' + 4y' + 4y = 0$

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0 \leftarrow$ Ouch! Only one root $r = -2$.

Only get one sol^n: $y_1 = e^{-2x}$.

$$L[e^{rx}] = (r^2 + 4r + 4)e^{rx} = (r+2)^2 e^{rx}$$

Euler: Take $\frac{\partial}{\partial r}$ of both sides. Because of constant coeff, can change order of derivatives: Get

$$\frac{\partial}{\partial r} L[e^{rx}] = L\left[\frac{\partial}{\partial r} e^{rx}\right] = \frac{\partial}{\partial r}\left((r+2)^2 e^{rx}\right)$$

$$L[xe^{rx}] = 2(r+2)e^{rx} + (r+2)^2 x e^{rx}$$

Aha! Now set $r = -2$:

$$L[xe^{-2x}] = 0$$

Got a second sol^n $y_2 = xe^{-2x}$

$$W[y_1, y_2] = \det\begin{bmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{bmatrix} = \det \begin{array}{cc} & \text{Col}_2 - x\,\text{Col}_1 \\ & \downarrow \\ \left[\begin{matrix} e^{-2x} \\ -2e^{-2x} \end{matrix}\right. & \left.\begin{matrix} 0 \\ e^{-2x} \end{matrix}\right] \end{array} = e^{-4x}$$

never zero. So

$$y = c_1 e^{-2x} + c_2 x e^{-2x}$$ is the gen^l sol^n!

Cool thing: Abel's theorem. The Wronskian is either $\equiv 0$ (in which case $y_1 = c y_2$) or never zero (in which case $c_1 y_1 + c_2 y_2$ is gen^l sol^n).

Note: y_1, y_2 are assumed to solve

$$\begin{cases} y_1'' + P y_1' + Q y_1 = 0 \leftarrow \text{mult by } y_2 \\ y_2'' + P y_2' + Q y_2 = 0 \leftarrow \text{mult by } y_1 \end{cases}$$

Bottom − Top:

$$\underbrace{(y_1 y_2'' - y_2 y_1'')}_{\frac{dW}{dx}} + P \underbrace{(y_1 y_2' - y_2 y_1')}_{W} = 0$$

$\leftarrow$ check that

$$\frac{dW}{dx} + PW = 0 \leftarrow \text{Linear!}$$

Get $\boxed{W = C e^{-\int P dx}}$ $\leftarrow$ Abel's formula

Aha! $C=0$ makes $W \equiv 0$. $C \neq 0$, then W never zero.

Consequence: Only need to check Wronskian at one point.

$$\frac{d}{dx} W = \frac{d}{dx} \det[\quad] = \frac{d}{dx}\left(y_1 y_2' - y_2 y_1'\right)$$

$$= y_1' y_2' + y_1 y_2'' - \left(y_2' y_1' + y_2 y_1''\right)$$

$$= y_1 y_2'' - y_2 y_1'' \checkmark$$

Roots: r_1, r_2 real and unequal : $b^2 - 4ac > 0$

r_1, r_2 real and same : $b^2 - 4ac = 0$

r_1, r_2 complex : $b^2 - 4ac < 0$

then what?

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = A \pm Bi \quad \text{complex}$$

Euler: Pretend not to notice!

For $r = A + Bi$. Got $y = e^{(A+Bi)x} = e^{Ax + (Bx)i}$

$$= e^{Ax} e^{(Bx)i}$$

$$= e^{Ax}\left(\cos Bx + i \sin Bx\right)$$

$$= \left(e^{Ax} \cos Bx\right) + i\left(e^{Ax} \sin Bx\right)$$

Euler's identity

$e^{i\theta} = \cos\theta + i \sin\theta$

$u + iv$ is a complex sol^n:

$$0 = L[u+iv] = \underbrace{L[u]}_{= 0} + i\,\underbrace{L[v]}_{= 0}$$

Aha! One complex sol^n contains two real sol^{n}s.

Cool thing: $e^{Ax}\cos Bx$ and $e^{Ax}\sin Bx$ are two real sol^{n}s and

$$W = \det\begin{bmatrix} e^{Ax}\cos Bx & e^{Ax}\sin Bx \\ Ae^{Ax}\cos Bx - Be^{Ax}\sin Bx & Ae^{Ax}\sin Bx + Be^{Ax}\cos Bx \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \end{array}$$

$$= \det\begin{bmatrix} e^{Ax}\cos Bx & e^{Ax}\sin Bx \\ -Be^{Ax}\sin Bx & Be^{Ax}\cos Bx \end{bmatrix} \begin{array}{l} \\ \leftarrow R_2 - AR_1 \end{array}$$

$$= Be^{Ax}\cdot e^{Ax}\underbrace{\det\begin{bmatrix} \cos Bx & \sin Bx \\ -\sin Bx & \cos Bx \end{bmatrix}}_{1} = \underset{\uparrow \text{ not zero}}{B}e^{2Ax}$$

So $y = c_1 e^{Ax}\cos Bx + c_2 e^{Ax}\sin Bx$ is gen^l sol^n

Euler: $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

$$e^{x+iy} = e^x e^{iy} = (e^x\cos y) + i(e^x \sin y)$$

$$e^{iy} = \left(1 + (iy) + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots\right)$$

$$= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \cdots$$

$$= \underbrace{\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right)}_{\cos y} + i\underbrace{\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots\right)}_{\sin y}$$

Fact: Complex does everything we want it to do!

$$u + iv = e^{(A+Bi)x}$$

$$\frac{d}{dx}\left[u+iv\right] = u' + iv' = (A+Bi)\,e^{(A+Bi)x}$$

So
$$L\left[e^{(A+Bi)x}\right] = \underbrace{\begin{pmatrix}\text{Char. Poly} \\ \text{with } r = A+Bi\end{pmatrix}}_{=0}\, e^{(A+Bi)x}$$

when $A+Bi$ is a root.

Conclusion: $e^{(A+Bi)x}$ is a complex sol^n.

From one complex sol^n, get two real sol^{n}s.

Hmmm. What do I get from $e^{(A-Bi)x}$?

$$y_1 = e^{Ax}\cos Bx \qquad y_2 = -e^{Ax}\sin Bx$$

Get same set of sol^{n}s!