Lesson 13 Complex roots

HWK 5 due Wed.

2nd order linear homog ODE with constant coeff

$$L[y] = ay'' + by' + cy = 0$$

want

Try $y = e^{rx}$. $L[e^{rx}] = (\underbrace{ar^2 + br + c}_{\text{need } =0})e^{rx} = 0$

Char. eqn.

Roots of $ar^2 + br + c = 0$: $r_1, r_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Case r_1, r_2 real and $\neq$: $(b^2 - 4ac > 0)$

Gen^l Sol^n: $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Case $r_1 = r_2$: $(b^2 - 4ac = 0)$

Gen^l Sol^n: $y = c_1 e^{r_1 x} + c_2 \underbrace{x e^{r_1 x}}$

second sol^n

$W \neq 0$.

Case $r = A \pm Bi$: $(b^2 - 4ac < 0)$

Gen^l Sol^n: $y = c_1 e^{Ax} \cos Bx + c_2 e^{Ax} \sin Bx$

Why the complex case works:

Complex sol^n: $y(x) = u(x) + i v(x)$

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h} = \lim_{h \to 0} \left[\frac{u(x+h) - u(x)}{h} + i \, \frac{v(x+h) - v(x)}{h} \right]$$

$$= u'(x) + i v'(x)$$

EX:

$$\frac{d}{dx}\Big[\underbrace{\cos x + i\sin x}_{e^{ix}}\Big] = \frac{d}{dx}\cos x + i\frac{d}{dx}\sin x$$

$$= -\sin x + i\cos x$$

$$= i(\cos x + i\sin x) = i e^{ix}$$

Fact:

$$\frac{d}{dx}\left[e^{(A+Bi)x}\right] = (A+Bi)\,e^{(A+Bi)x}$$

(Write out both sides. Expand. See =)

$$\frac{d^2}{dx^2}\left[e^{(A+Bi)x}\right] = (A+Bi)^2 e^{(A+Bi)x} \quad \text{too!}$$

Claim:

$$e^{(A+Bi)x} = \underbrace{u(x)}_{u(x)=e^{Ax}\cos Bx} + i\,\underbrace{v(x)}_{v(x)=e^{Ax}\sin Bx}$$

If $r = A+Bi$ is a root of Char. eqn, then

$e^{(A+Bi)x}$ is a complex sol^n.

Complex sol^n means: $\quad Y = u + iv$

$$aY'' + bY' + cY = 0$$

$$a(u'' + iv'') + b(u' + iv') + c(u + iv) = 0$$

$$\underbrace{\left[au'' + bu' + cu\right]}_{=0} + i\underbrace{\left[av'' + bv' + cv\right]}_{=0} = 0 + 0i$$

Big fact: One complex sol^n $\Psi = u + iv$ gives us two real sol^{n}s: u, v.

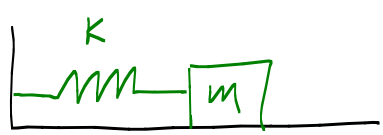
Why is $e^{(A+Bi)x}$ a complex sol^n: Let $r = A+Bi$.

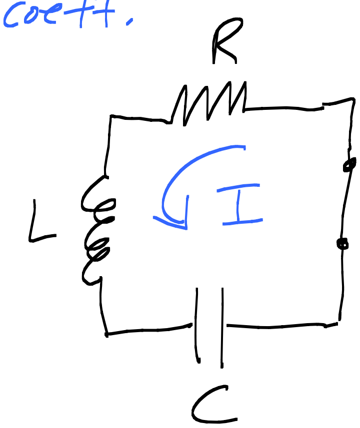
We chose r to be a root of $ar^2 + br + c = 0$.

$$L[e^{rx}] = a\frac{d^2}{dx^2}(e^{rx}) + b\frac{d}{dx}(e^{rx}) + ce^{rx}$$

$$= (\underbrace{ar^2 + br + c}_{=0})e^{rx} = 0$$

Important case: positive coeff.





$$F = ma$$

$$-ky - cy' = my''$$

$$my'' + cy' + ky = 0$$

$$LI'' + RI' + \frac{1}{C}I = 0$$

$ar^2 + br + c = 0$ a, b, c positive

Case r_1, r_2 real $\neq$. $b^2 - 4ac > 0$

$$0 < b^2 - 4ac < b^2$$

So $$0 < \sqrt{b^2 - 4ac} < b$$

Aha! $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ is negative.

$\dfrac{-b-\sqrt{b^2-4ac}}{2a}$ is obviously negative.

r_1, r_2 are negative!

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} \longrightarrow 0 \text{ as } x \to \infty.$$

Sol^{n}s die out. Case: Over damped

Case $r_1 = r_2$: $r_1 = \dfrac{-b}{2a}$ ← negative!

$$y = c_1 \underbrace{e^{-\frac{b}{2a}x}}_{\substack{\to 0 \\ \text{as } x\to\infty}} + c_2 \underbrace{x e^{-\frac{b}{2a}x}}_{\substack{\to 0 \text{ too!} \\ \text{as } x\to\infty}} \to 0 \text{ as } x \to \infty.$$

Critically damped

Case complex roots:

$$r_1, r_2 = \frac{-b}{2a} \pm \frac{\sqrt{neg}}{2a}$$

$$= \frac{-b}{2a} + Bi$$

$$y = c_1 e^{-\frac{b}{2a}x} \cos Bx + c_2 e^{-\frac{b}{2a}x} \sin Bx$$

$$\longrightarrow 0 \text{ as } x \to \infty.$$

Under damped.