Lesson 14 Reduction of order (repeated roots) 3.4 HWK6 due Wed.

Hmmm. $\left(\frac{y_2}{y_1}\right)' = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \frac{W}{y_1^2} = \frac{c\, e^{-\int P(x)dx}}{y_1^2}$

How odd! No y_2 on RHS!

What if only know one sol^n y_1. Want y_2.

$$\left(\frac{y_2}{y_1}\right)' = c\,\frac{e^{-\int P dx}}{y_1^2}$$

$$\frac{y_2}{y_1} = \int \sim\, dx = v(x)$$

Then $y_2 = v\,y_1$.

Method of reduction of order: $A(x)y'' + B(x)y' + C(x)y = 0$

Put in Standard Form: $\boxed{y'' + P(x)y' + Q(x)y = 0}$

Suppose we only know one sol^n $y_1(x)$.

Look for a second sol^n of the form $\boxed{y_2 = v\,y_1}$.

Coming soon: A formula for v:

$$\boxed{v' = \frac{1}{y_1^2}\, e^{-\int P(x)dx}}$$

Calculate RHS. Integrate to get v. [Take $C=0$].

Then $y_2 = v\,y_1$.

<u>Why it works</u>: $L[y] = y'' + P(x)y' + Q(x)y$

We have y_1: $L[y_1] = 0$

We want a y_2. Look for $y_2 = v\,y_1$ ← y_2 of this form

<u>Step 1</u>: Plug vy_1 into ODE and boss it around.

$$(vy_1)'' + P(vy_1)' + Q(vy_1) \overset{\text{want}}{=} 0$$

$$v''y_1 + 2v'y_1' + vy_1'' + Pv'y_1 + Pvy_1' + Qvy_1 = 0$$

$$v\,\underbrace{L[y_1]}_{=0} =$$

$$v''y_1 + 2v'y_1' + Pv'y_1 = 0$$

$$\boxed{v'' + \left(\frac{2y_1'}{y_1} + P\right)v' = 0}$$

Can solve this! It's a linear first order ODE in $u = v'$:

"reduction of order"

$$\mu\left(\frac{du}{dx} + \left(\frac{2y_1'}{y_1} + P\right)u\right) = 0$$

$$[\mu u]' = 0 \qquad \mu u = C$$

Int. factor: $\mu = e^{\int \frac{2y_1'}{y_1} + P \, dx}$

$$= e^{\int 2\frac{d}{dx}\ln|y_1| + P \, dx}$$

$$= e^{2\ln|y_1| + \int P \, dx}$$

$$= e^{\ln y_1^2} \; e^{\int P \, dx} = y_1^2 \, e^{\int P dx}$$

Use μ: $\mu\,(\text{ODE}) = 0$

$$[\mu u]' = 0$$

$$\mu u = C$$

$$u = \frac{C}{\mu} \leftarrow \text{Take } C = 1$$

$$u = \frac{1}{\mu} = \frac{1}{y_1^2} e^{-\int P(x)\,dx}$$

Done! $$\boxed{v' = \frac{1}{y_1^2} e^{-\int P(x) dx}}$$

EX: $y'' + \underline{\underline{4}}y' + 4y = 0 \leftarrow$ stand Form!

So $\underline{\underline{P(x) = 4}}$

Try e^{rx}. Need $r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$ Repeated root!

$\boxed{r = -2.}$ Only get one sol^{n}

$$\boxed{y_1 = e^{-2x}}$$

Book: Try $y_2 = v\,e^{-2x}$.

Plug into ODE. Get new ODE

$$\frac{du}{dx} + (\text{eee})u = 0$$

where $u = v'$.

My way: $y_2 = v\,y_1$ where $\boxed{v' = \frac{1}{y_1^2} e^{-\int P(x)\,dx}}$

$$v' = \frac{1}{(e^{-2x})^2} e^{-\int 4\,dx}$$

$$= \frac{1}{e^{-4x}} e^{-4x} = 1$$

Get $\boxed{v' = 1}$. Integrate: $v = \int 1\,dx = x$ ___

No $+C$ here

I want $\underline{a}$ 2^{nd} sol'n, not all possible.

Aha! Got $y_2 = v y_1 = x e^{-2x}$

just like Euler did!

<u>Euler eqns</u>: $a x^2 y'' + b x y' + c y = 0$ ← not const coeff!

Euler: guess $y = x^r$

$$y' = r x^{r-1}$$

$$y'' = r(r-1)x^{r-2}$$

Plug in and force it.

$$ax^2y'' + bxy' + cy = 0$$

$$ax^2\left[r(r-1)x^{r-2}\right] + bx\left[rx^{r-1}\right] + cx^r = \underset{\text{want}}{0}$$

$$\Big[\underbrace{ar(r-1) + br + c}_{\text{need } = 0}\Big]x^r = 0$$

Good case: Get $r_1, r_2 \neq$ real. $W[x^{r_1}, x^{r_2}] \neq 0$

Then $y = c_1 x^{r_1} + c_2 x^{r_2}$ is gen^l sol^n.

EX: $x^2y'' + 2xy' - 2y = 0$

$$r(r-1) + 2r - 2 = 0$$

$$r^2 + r - 2 = 0$$

$$(r+2)(r-1) = 0 \quad \text{Roots } r = 1, -2.$$

Gen^l sol^n: $\boxed{y = c_1 x + c_2 x^{-2}}$

What if roots are repeated?

EX: $x^2y'' + 3xy' + y = 0$

$$\boxed{y'' + \frac{3}{x}y' + \frac{1}{x^2}y = 0}$$

S. Form

$P(x)$

$$r(r-1) + 3r + 1 = 0$$

$$r^2 + 2r + 1$$

$$(r+1)^2 = 0 \quad \text{Repeated: } r = -1, -1$$

Only get $y_1 = x^{-1} = \frac{1}{x}$

Get $y_2 = v\, y_1$ where

$$v' = \frac{1}{y_1^2} e^{-\int P(x)\,dx}$$

$$v' = \frac{1}{\left(\frac{1}{x}\right)^2} e^{-\int \frac{3}{x}\,dx}$$ ← assume $x > 0$

$$= x^2 e^{-3\ln x} = x^2 e^{\ln x^{-3}} = x^2 \cdot x^{-3} = x^{-1}$$

$$\boxed{v' = \frac{1}{x}}$$

So $v = \int \frac{1}{x}\,dx = \ln x$

Get $y_2 = (\ln x) \cdot x^{-1}$

Could check $W[y_1, y_2]$. see $\neq 0$. Guaranteed though by method.

Gen^l Sol^n: $y = c_1 \cdot \frac{1}{x} + c_2 \frac{\ln x}{x}$

Euler eqns: Repeated root case: $y_1 = x^{r_1}$

$y_2 = (\ln x)\, x^{r_1}$

What about $r =$ complex root case?

$$y = x^{a+bi} = \left(e^{\ln x}\right)^{a+bi}$$ ← This is a complex sol^n!

$$= e^{(a+bi)\ln x} = \underbrace{e^{a\ln x}}_{e^{\ln x^a}} \underbrace{e^{(b\ln x)i}}_{\cos(b\ln x) + i\sin(b\ln x)}$$

$$= \underbrace{x^a \cos(b\ln x)}_{y_1} + i\, \underbrace{x^a \sin(b\ln x)}_{y_2}$$

From one complex sol^n, get two real sol^{n}s.

$$W[y_1, y_2] \neq 0$$

$y = c_1 y_1 + c_2 y_2$ is gen^l sol^n.