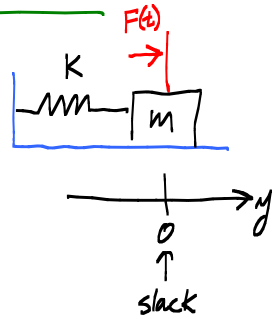
Lesson 15 Non-homogeneous 2nd order linear eqns, Method of undetermined coefficients



$$F = ma$$

$$F(t) - ky - cy' = my''$$

↑ external force

$$my'' + cy' + ky = \cancel{0}\ F(t)$$ ← non-homogeneous

$$mr^2 + cr + k = 0$$

roots:

Overdamped → r_1, r_2 real $\neq$, both < 0 $\quad e^{r_1 t}, e^{r_2 t} \to 0$ as $t \to \infty$

Critically damped → r_1 real $=$, $r_1 < 0$ $\quad e^{r_1 t}, te^{r_1 t} \to 0$ as $t \to \infty$

Underdamped → r_1, r_2 complex, $\underbrace{\text{Real part}}_{-A} < 0$ $\quad e^{-At}\cos Bt, e^{-At}\sin Bt \to 0$ as $t \to \infty$

Above: $c \neq 0$. Then all sol'ns $\to 0$ as $t \to \infty$.

↑ friction coeff

What if $c = 0$?

$$my'' + ky = 0$$

$$mr^2 + k = 0$$

$$r^2 = -\frac{k}{m}$$

$$r = \pm i\sqrt{\frac{k}{m}} \qquad \text{Let } \omega = \sqrt{\frac{k}{m}}$$

Roots: $r_1, r_2 = \pm i\omega = 0 \pm i\omega$

Two sol'ns $\quad y_1 = e^{0 \cdot t}\cos\omega t, \quad y_2 = e^{0 \cdot t}\sin\omega t$

Gen'l Sol'n: $\quad y = c_1\cos\omega t + c_2\sin\omega t$

Undamped case

$$= A\cos(\omega t - \varphi)$$

Simple harmonic motion. No friction $\Rightarrow$ sol'ns $\not\to 0$.

Inhomogeneous 2^{nd} order linear ODE $A(x)y'' + B(x)y' + C(x)y = D(x)$

Standard Form: $y'' + P(x)y' + Q(x)y = F(x)$ $\quad (+)$

Associated homogeneous eqn: $y'' + Py' + Qy = 0$ $\quad (*)$

Big fact: General sol^n to $(+)$ is

$$\boxed{y = y_p + (c_1 y_1 + c_2 y_2)}$$

where y_p is one particular sol^n to $(+)$

and $c_1 y_1 + c_2 y_2$ is the gen' sol^n to homog eqn $(*)$.

Why: Write $L[y] = y'' + Py' + Qy$.

Suppose $L[y_p] = F$ $\leftarrow$ y_p is a part sol^n.

If Y also solves $L[Y] = F$, then

$$L[\underbrace{Y - y_p}] = L[Y] - L[y_p] = F - F = 0$$

Aha! This solves $(*)$.

So $Y - y_p = c_1 y_1 + c_2 y_2$ for some choice of c_1, c_2.

So $Y = y_p + (c_1 y_1 + c_2 y_2)$ ✓

Not quite done. Need to show that anything of the form $y_p + (c_1 y_1 + c_2 y_2)$ solves $(+)$ too.

Easy! $L[\underline{y_p + (c_1 y_1 + c_2 y_2)}] = \underbrace{L[y_p]}_{F} + c_1 \underbrace{L[y_1]}_{0} + c_2 \underbrace{L[y_2]}_{0}$

$= F$ ✓

EX: $y'' + y = e^{-2x}$

Step 1: Solve homog eqn: $y'' + y = 0$

$r^2 + 1 = 0$

$r = \pm i$

Gen^l Sol^n to (*) is $c_1 \cos x + c_2 \sin x$

Step 2: Find one stupid sol^n to (+):

$$y'' + y = e^{-2x}$$

Euler: Guess $y_p = Ae^{-2x}$

$y_p' = -2Ae^{-2x}$

$y_p'' = 4Ae^{-2x}$

Want $(y_p)'' + y_p = e^{-2x}$

$4Ae^{-2x} + Ae^{-2x} = e^{-2x}$ want

$5Ae^{-2x} = e^{-2x}$

$\boxed{5A = 1}$ $A = 1/5$

Get $y_p = \frac{1}{5} e^{-2x}$

Step 3: Gen^l Sol^n to (+)

$$y = \frac{1}{5}e^{-2x} + (c_1 \cos x + c_2 \sin x)$$

Step 4: I.V.P. $y(0) = A$: $\frac{1}{5} + c_1 \cdot 1 + c_2 \cdot 0 = A$

$y'(0) = B$ $\quad -2 \cdot \frac{1}{5} - c_1 \cdot 0 + c_2 \cdot 1 = B$

want

$$\begin{cases} c_1 = A - \frac{1}{5} \\ c_2 = B + \frac{2}{5} \end{cases}$$

E & U Thm: P, Q, F continuous on (a,b) containing x_0, then a sol^n to IVP at x_0 exists and is unique and valid on (a,b).

Method of undetermined coefficients

$$\underbrace{ay'' + by' + cy}_{\text{2nd order linear, const coeff}} = F(x)$$

special form

$F(x)$	Guess $y_p =$
e^{rx}	$y_p = Ae^{rx}$
x^n (or any poly of deg n)	$y_p = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$
$\cos \omega x$	$y_p = A \cos \omega x + B \sin \omega x$

$\sin \omega x$	same
$e^{rx}\cos\omega x$ or $e^{rx}\sin\omega x$	$y_p = Ae^{rx}\cos\omega x + Be^{rx}\sin\omega x$
$x^n e^{rx}\cos\omega x$ or $x^n e^{rx}\sin\omega x$	$y_p = (A_n x^n + \dots + A_0)e^{rx}\cos\omega x + (B_n x^n + \dots + B_0)e^{rx}\sin\omega x$

Important Rule: But, if any single piece (like xe^{2x}) solves the homogeneous problem, multiply the guess sol^n by x^m where m is the smallest positive integer that eliminates this issue.

Famous problem: $y'' + y = \sin x$

Homog sol^n: $c_1 \cos x + c_2 \sin x$

Table: Try $y_p = A\cos x + B\sin x \leftarrow$ Ouch! These solve homog. $L[y_p] \equiv 0 \neq \sin x$

Rule: Correct try $y_p = x\,(A\cos x + B\sin x)$

$$= A\underbrace{(x\cos x)}_{\text{Don't solve (x)}} + B\underbrace{(x\sin x)}_{\text{Don't solve (x)}} \checkmark$$

$$y_p' = (A\cos x + B\sin x) + x(-A\sin x + B\cos x)$$

$$y_p'' = -A\sin x + B\cos x - A\sin x + B\cos x + x(-A\cos x - B\sin x)$$

$$y_p'' = 2B\cos x - 2A\sin x + x(-A\cos x - B\sin x)$$

Plug in: $$y_p'' + y_p = 2B\cos x - 2A\sin x \overset{\text{want}}{=} \sin x$$

Need $2B = 0$ $\quad B = 0$

$-2A = 1$ $\quad A = -\frac{1}{2}$

Get $$y_p = x(A\cos x + B\sin x) = -\frac{1}{2}x\cos x$$

Resonance! $$\underbrace{y'' + y}_{\omega = 1 \text{ is the natural freq}} = \underbrace{\sin x}_{\text{forcing with natural freq}}$$

y_p