Lesson 16 Variation of parameters

HWK 6 due Wed

EX: $y'' + 2y' + y = e^{-x}$

Step 1: Homogenous eqn $y'' + 2y' + y = 0$

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0$$

Roots $r = -1, -1$ repeated

$$y_c = \underline{c_1 e^{-x} + c_2 x e^{-x}}$$

Step 2: Find one particular sol^n y_p.

Use Method of coeff: From table: Try $y_p = \underline{\underline{Ae^{-x}}}$

ouch! Solves homog.

Rule: Multiply by x until problem is cleared:

Next try: $y_p = A\underline{\underline{xe^{-x}}}$

Ouch again!

Finally try: $\boxed{y_p = Ax^2e^{-x}}$ ← This works! Plug into ODE. Determine A.

Step 3: Gen^l Sol^n: $c_1 e^{-x} + c_2 x e^{-x} + \underline{\underline{A}}x^2 e^{-x}$

your A goes here

Variation of parameters:

$$A(x)y'' + B(x)y' + C(x)y = D(x)$$

Step 1: Put in Standard Form: $y'' + P(x)y' + Q(x)y = F(x) \quad (+)$

Assume we know gen'l sol'n to homog eqn

$$y'' + Py' + Qy = 0$$

is $\quad y = c_1 y_1 + c_2 y_2$

Big idea: Look for a particular sol'n of form

$$y_p = u_1 y_1 + u_2 y_2$$

where u_1 and u_2 are unknown fcns determined by plugging into non-homog eqn $(+)$.

Get
$$\begin{cases} u_1' = \dfrac{-y_2 F}{W[y_1, y_2]} \\[2ex] u_2' = \dfrac{y_1 F}{W[y_1, y_2]} \end{cases}$$

$\leftarrow$ F = RHS in Standard Form

Integrate to get u_1 and u_2.

Gen'l Sol'n to $(+)$ is $\underbrace{(c_1 y_1 + c_2 y_2)}_{y_c} + \underbrace{(u_1 y_1 + u_2 y_2)}_{y_p}$

Why it works: Try $\quad y_p = u_1 y_1 + u_2 y_2$

Plug into ODE and force it. $\leftarrow$ only gives one eqn for two fcns!

$$y_p' = \underbrace{u_1' y_1 + u_2' y_2}_{=0} + u_1 y_1' + u_2 y_2'$$

Aha! $\quad = 0 \quad$ let this be the other eqn!

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

Plug into ODE:

$$\left[u_1' y_1' + u_2' y_2' + \underline{\underline{u_1 y_1''}} + \underline{\underline{u_2 y_2''}}\right] + P\left[\underline{\underline{u_1 y_1'}} + \underline{\underline{u_2 y_2'}}\right] + Q\left[\underline{\underline{u_1 y_1}} + \underline{\underline{u_2 y_2}}\right] \overset{\text{want}}{=} F$$

Blue things: $u_1 L[y_1] = 0$

Green things: $u_2 L[y_2] = 0$

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 & \leftarrow \text{gave ourselves} \\ u_1' y_1' + u_2' y_2' = F & \leftarrow \text{from ODE} \end{cases}$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix}$$

Cramer's rule!

$$u_1' = \frac{\det \begin{bmatrix} 0 & y_2 \\ F & y_2' \end{bmatrix}}{\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}} = \frac{-y_2 F}{W}$$

$\leftarrow W!$

$$u_2' = \frac{\det \begin{bmatrix} y_1 & 0 \\ y_1' & F \end{bmatrix}}{\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}} = \frac{y_1 F}{W}$$

EX: $y'' + y = \underbrace{\sec x}$ not special! bad!

1 in front ✓

Standard Form. So $F(x) = \sec x$.

Step 1: Homog solⁿ: $r = \pm i = 0 \pm i$

$$y_c = c_1 \underbrace{\cos x}_{y_1} + c_2 \underbrace{\sin x}_{y_2}$$

Step 2: $W[y_1, y_2] = \det \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$

$$= \cos^2 x + \sin^2 x \equiv 1$$

Step 3:

$$u_1' = \frac{-y_2 F}{W} = \frac{-(\sin x)\sec x}{1}$$

$\sec = \frac{1}{\cos}$

$$= -\tan x$$

$$u_2' = \frac{y_1 F}{W} = \frac{\cos x \sec x}{1} = 1$$

Step 3: Integrate

$$u_1 = \int -\tan x \, dx = -\ln|\sec x|$$

skip +C Just want one y_p

$$= \ln|\sec x|^{-1} = \ln|\cos x|$$

$$u_2 = \int 1 \, dx = x$$

no +C

Step 4: $y_p = u_1 y_1 + u_2 y_2$

$$= \underbrace{\left(\ln|\cos x|\right)\left(\cos x\right) + x \sin x}_{y_p}$$

Step 5: Gen^l Sol^n to (+)

$$y = c_1 \cos x + c_2 \sin x + y_p$$

EX: $x^2 y'' - 4x y' + 4y = -x^2$

Step 1: Homog eqn: $x^2 y'' - 4x y' + 4y = 0$

Euler eqn: Try $y = x^r$

$$\left[\underbrace{r(r-1) - 4r + 4}_{=0}\right] x^r \overset{\text{want}}{=} 0$$

Roots $r = 1, 4$

Homog sol^n $y = c_1 \underbrace{x}_{y_1} + c_2 \underbrace{x^4}_{y_2}$

Step 2: $W = \det \begin{bmatrix} x & x^4 \\ 1 & 4x^3 \end{bmatrix} = 4x^4 - x^4 = 3x^4$

Danger: $x^2 y'' - 4x y' + 4y = -x^2$ Not in Stand Form.

$$y'' - \frac{4}{x} y' + \frac{4}{x^2} y = -\underline{1} \leftarrow F(x)!$$

[$x=0$ is bad! Explains why $W(0)=0$.]

Step:
$$u_1' = \frac{-(x^4)(-1)}{3x^4} = \frac{1}{3}$$

$$u_2' = \frac{(x)(-1)}{3x^4} = -\frac{1}{3x^3}$$

Step:
$$u_1 = \int \frac{1}{3}\,dx = \frac{1}{3}x$$

$$u_2 = \int \frac{-1}{3x^3}\,dx = \frac{1}{6}x^{-2}$$

Sol^n:
$$\left(c_1 x + c_2 x^4\right) + \left(\underbrace{\left(\tfrac{1}{3}x\right)x + \left(\tfrac{1}{6}x^{-2}\right)x^4}_{\frac{1}{3}x^2 + \frac{1}{6}x^2 = \underbrace{\frac{1}{2}x^2}_{y_p}}\right)$$

Important fact:
$$A(x)y'' + B(x)y' + C(x)y = F_1(x) + F_2(x) + F_3(x)$$

Good news: $L[y] = Ay'' + By' + Cy$ is linear.

Get y_{p_j} : $L[y_{p_j}] = F_j$

Then
$$L[\underbrace{y_{p_1} + y_{p_2} + y_{p_3}}_{y_p}] = L[y_{p_1}] + \cdots + L[y_{p_3}]$$
$$= F_1 + F_2 + F_3 \quad \checkmark$$

EX: $y'' + y = \sin x + x^2 + e^{2x}$

Form of y_p for undetermined coeff =

$$y_p = x\left(A\cos x + B\sin x\right) + \left(Cx^2 + Dx + E\right) + Fe^{2x}$$

Advice: Split into 3 problems: F_1, F_2, F_3

EX:

$$\underbrace{y'' + y'}_{y_c = c_1 e^{-x} + c_2} = x^2 \leftarrow$$

From table: $y_p = Ax^2 + Bx + C$

ouch!

Correct guess: $y_p = x\left(Ax^2 + Bx + C\right)$

Individual pieces: x^3, x^2, x safe!