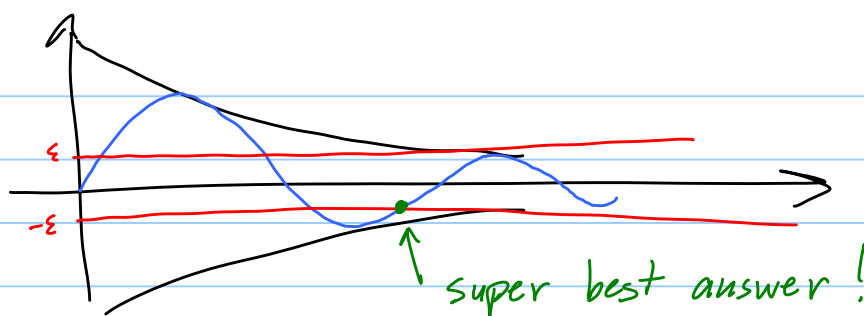


Lesson 18 : Forced vibrations



$$Ae^{-kt} \cos(\omega t - \phi)$$

Good answer: When is $|Ae^{-kt}| < \epsilon$?

Choice: $y = c_1 \cos \omega t + c_2 \sin \omega t$ or $A \cos(\omega t - \phi)$

$\uparrow \quad \quad \uparrow$ $\uparrow \quad \quad \uparrow$
 2 const's. 2 const's.

2 initial conditions: $\begin{cases} y(0) = y_0 \\ y'(0) = v_0 \end{cases}$ $\begin{cases} y(0) = A \cos(0 - \phi) = y_0 \\ y'(0) = -A\omega \sin(0 - \phi) = v_0 \end{cases}$ want

$$\begin{cases} A \cos \phi = y_0 \\ A\omega \sin \phi = v_0 \end{cases}$$

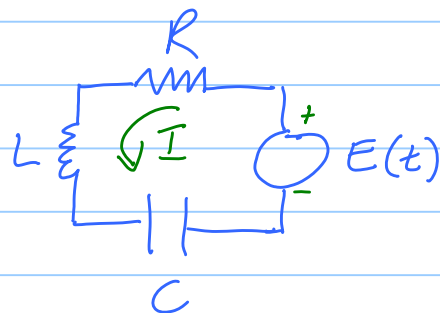
$$\frac{A\omega \sin \phi}{A \cos \phi} = \frac{v_0}{y_0}$$

$\tan \phi = \frac{v_0}{\omega y_0}$

Get ϕ

$$A = y_0 / \cos \phi$$

Electric circuits:



Kirchoff's Law:

Sum voltage drops around a loop = 0.

Resistor: Ohm's Law: IR

Inductor: $L \frac{dI}{dt}$

Capacitor: $\frac{1}{C} Q$

I = current

Q = charge

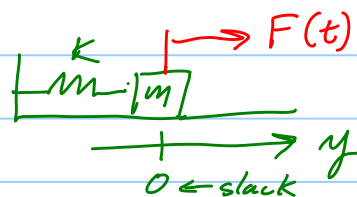
Kirchoff's: $L \frac{dI}{dt} + IR + \frac{1}{C} Q = E(t)$

$I = \frac{dQ}{dt}$!

Differentiate:

$$L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = \mathcal{E}'(t)$$

Forced vibrations: Undamped case:



$$m \ddot{y} + K y = F(t)$$

EX: $m \ddot{y} + K y = F_0 \cos \omega t$

Step 1: Solve homog: $m \ddot{y} + K y = 0$

$$m r^2 + K = 0 \quad r = \pm \sqrt{\frac{K}{m}} i$$

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

$$\omega_0 = \sqrt{\frac{K}{m}}$$

Step 2: Find a particular solⁿ:

Case $\omega \neq \omega_0$: Try $y_p = A \cos \omega t + B \sin \omega t$

$$y_p'' = -A \omega^2 \cos \omega t - B \omega^2 \sin \omega t$$

Plug in: $m [-A \omega^2 \cos \omega t - B \omega^2 \sin \omega t] + K [A \cos \omega t + B \sin \omega t] = F_0 \cos \omega t$ want

$$\underbrace{[-A m \omega^2 + K A]}_{F_0} \cos \omega t + \underbrace{[-B m \omega^2 + K B]}_0 \sin \omega t =$$

Get $B = 0$, $A = \frac{F_0}{K - m \omega^2} = \frac{F_0}{m \left(\frac{K}{m} - \omega^2 \right)} = \frac{F_0}{m (\omega_0^2 - \omega^2)}$ ω_0^2

Step 3: Gen^l Solⁿ:

$$y = \underbrace{c_1 \cos \omega_0 t + c_2 \sin \omega_0 t}_{A \cos(\omega_0 t - \phi)} + \frac{F_0}{m (\omega_0^2 - \omega^2)} \cos \omega t$$

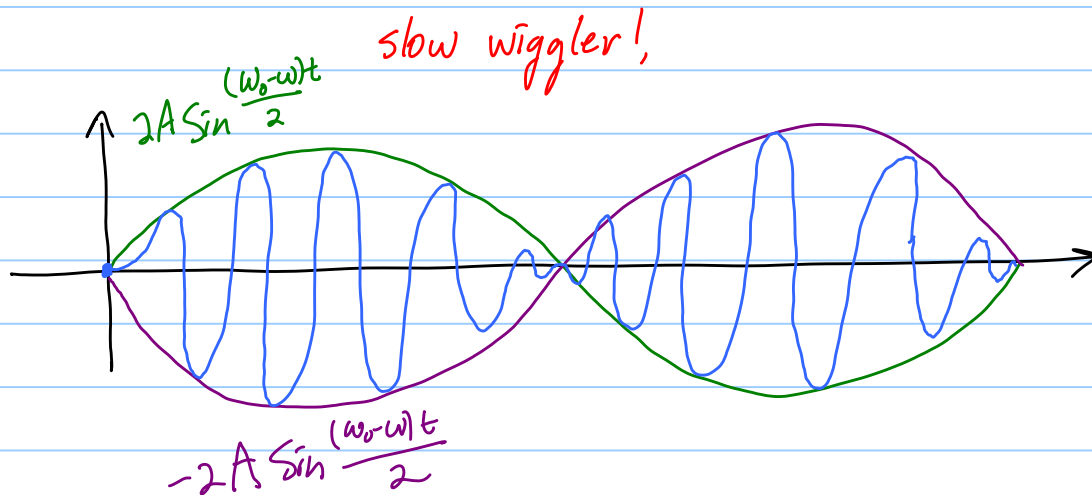
Whoa! $\omega \approx \omega_0$ causes huge vibrations

EX: $y = A (\cos \omega t - \cos \omega_0 t)$

$$= 2A \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2}$$

$\uparrow$ small $\uparrow$ close to ω_0

slow wiggler!



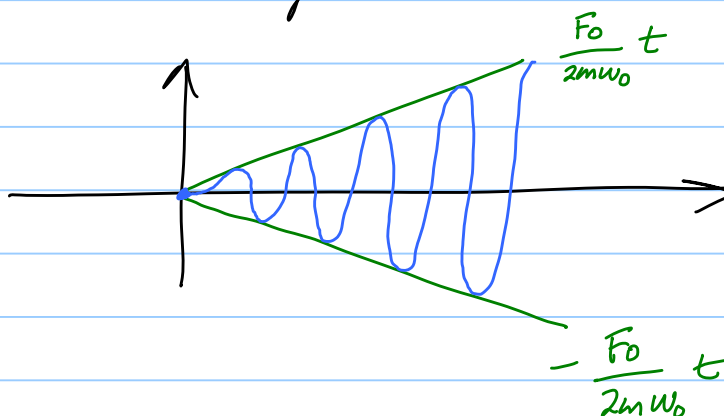
Case $\omega = \omega_0$: Resonance!

Try $y_p = t (A \cos \omega_0 t + B \sin \omega_0 t)$

EX: $m y'' + K y = F_0 \cos \omega_0 t$

Get $y_p = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$

Prob: Resonance. $\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$ $y = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$



Damping: $m\ddot{y} + c\dot{y} + ky = F_0 \cos \omega t$ Underdamped $c^2 - 4km < 0$

Step 1: homog: $r = -\frac{c}{2m} \pm \underbrace{\sqrt{c^2 - 4km}}_{\mu \leftarrow \text{quasi freq}} i$

$y = A e^{-\frac{c}{2m}t} \cos(\mu t - \phi)$ $\omega_0 = \sqrt{\frac{k}{m}}$

Step 2: Particular solⁿ $y_p = A \cos \omega t + B \sin \omega t$
 No danger!

Plug in. Get linear eqns in A, B. Solve.

$y_p = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}} \cos(\omega t - \eta)$

Words: $y = \underbrace{\left(\text{Homog sol}^n \right)}_{\substack{\text{dies out} \\ \text{because of neg.} \\ \text{exponentials}}} + \underbrace{\left[\text{particular sol}^n \right]}_{\substack{\text{does not} \\ \text{die away!}}}$
 "Transient" "Steady state"