Lesson 19 Higher order linear ODE 4.1, 4.2 HWK 7 due Friday

$$A_n(x)\frac{d^n y}{dx^n} + A_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + A_1(x)\frac{dy}{dx} + A_0(x)\,y = \begin{cases} 0 & \text{Homog} \\ F(x) & \text{Inhomg} \end{cases}$$

zeroes of $A_n(x)$ are dangerous!

EX: $y^{(n)} \equiv 0$

Integrate n-times: $y = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1}x^{n-1}$

Get n-constants, n-solutions: $1, x, x^2, \ldots, x^{n-1}$

Step 1: Put ODE in standard form: Divide by $A_n(x)$ =

$$\frac{d^n y}{dx^n} + P_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_1(x)\frac{dy}{dx} + P_0(x)\,y = \begin{cases} 0 & (*) \\ f(x) & (\dagger) \end{cases}$$

E & U Thm: The Initial Value Problem (IVP), y solving

(*) or (†) with $\begin{cases} y(x_0) = y_0 \\ y'(x_0) = y_0' \\ \vdots \\ y^{(n-1)}(x_0) = y_0^{(n-1)} \end{cases}$ ← n conditions.
Think: for n constants

has a unique sol^n when all the fcns $P_j(x)$ are

E U

continuous near x_0 (and $f(x)$ is cont for (†) too).

Sol^n guaranteed valid on largest interval containing x_0 where P's are all continuous.

Big fact Superposition principle holds:

$$L[y] = y^{(n)} + P_{n-1}\, y^{(n-1)} + \cdots + P_1 y' + P_0 y$$

is a linear operator. So linear combinations of sol^{n}s to the homog equ are sol^{n}s too.

Why: $L[c_1 y_1 + \cdots + c_n y_n] = c_1 \underbrace{L[y_1]}_{0} + \cdots + c_n \underbrace{L[y_n]}_{0} = 0$ ✓

EX: Constant coeff case:

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0 \qquad (*)$$

Try $y = e^{rx}$:

$$y' = re^{rx}$$
$$y'' = r^2 e^{rx}$$
$$\vdots$$
$$y^{(n)} = r^n e^{rx}$$

want

Plug into $(*)$: $a_n r^n e^{rx} + \cdots + a_0 e^{rx} = 0$

$$\underbrace{\left(a_n r^n + \cdots + a_1 r + a_0\right)}_{\text{Need this } = 0} e^{rx} = 0$$

never zero

"Characteristic equ"

Hope: Get n distinct real roots $r_1, r_2, \ldots, r_n$.

Get n sol^{n}s $e^{r_j x}$

Hope: $y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$ is gen^{l} sol^{n}.

Ex: 3rd order prob: Get $r = 2, 3, 5$

$$y_1 = e^{2x}, \quad y_2 = e^{3x}, \quad y_3 = e^{5x}$$

Is $y = c_1 e^{2x} + c_2 e^{3x} + c_3 e^{5x}$ the gen'l sol'n?

Equiv question: Can I solve <u>any</u> and <u>all</u> IVPs with this sol'n?

IVP
$$\begin{cases} y(0) = c_1 + c_2 + c_3 = A \\ y'(0) = 2c_1 + 3c_2 + 5c_3 = B \\ y''(0) = 4c_1 + 9c_2 + 25c_3 = C \end{cases}$$

given

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 9 & 25 \end{bmatrix}} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

Need $\det \neq 0$!

How to remember linear alg fact: Cramer's rule:

$$c_2 = \frac{\det \begin{bmatrix} 1 & A & 1 \\ 2 & B & 5 \\ 4 & C & 25 \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 9 & 25 \end{bmatrix}}$$

need $\neq 0$

<u>Def'n</u>: $W[y_1, y_2, y_3] = \det \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix}$

$$W[y_1, \dots, y_n] = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$$

<u>Fact</u>: If $W(0) \neq 0$, get general sol^n for $(*)$.

<u>EX</u>:

$$\det \begin{bmatrix} e^{r_1 x} & e^{r_2 x} & e^{r_3 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} & r_3 e^{r_3 x} \\ r_1^2 e^{r_1} & r_2^2 e^{r_2 x} & r_3^2 e^{r_3 x} \end{bmatrix} \begin{matrix} R1 \\ R2 \\ R3 \end{matrix}$$

$$= \det \begin{bmatrix} e^{r_1 x} & e^{r_2 x} & e^{r_3 x} \\ 0 & (r_2 - r_1) e^{r_2 x} & (r_3 - r_1) e^{r_3 x} \\ 0 & (r_2^2 - r_1^2) e^{r_2 x} & (r_3^2 - r_1^2) e^{r_3 x} \end{bmatrix} \begin{matrix} \\ R2 - r_1 R1 \\ R3 - r_1^2 R1 \end{matrix}$$

$$= \det \begin{bmatrix} e^{r_1 x} & \ell\ell\ell & \ell\ell\ell \\ 0 & (r_2 - r_1) e^{r_2 x} & (r_3 - r_1) e^{r_3 x} \\ 0 & 0 & * \end{bmatrix} \begin{matrix} \\ \\ R_3 - (r_2 + r_1) R2 \end{matrix}$$

$$* = \left[(r_3^2 - r_1^2) - (r_2 + r_1)(r_3 - r_1)\right] e^{r_3 x}$$

$$= e^{r_1 x} \cdot (r_2 - r_1) e^{r_2 x} \cdot \left[(r_3 - r_1)\left(r_3 + r_1 - (r_2 + r_1)\right)\right] e^{r_3 x}$$

$$= (r_2 - r_1)(r_3 - r_1)(r_3 - r_2)\, e^{(r_1 + r_2 + r_3) x}$$

Never zero if r's are distinct.

Fact: Abel's formula: Either $W \equiv 0$ or never zero.

So just checking W at one pt is enough.

Big fact: The gen^l sol^n to $(+)$ ← non-homog prob

is

$$\underbrace{\left[c_1 y_1 + c_2 y_2 + \cdots + c_n y_n\right]}_{\text{gen}^l\ \text{sol}^n\ \text{to homog prob } (*)} + \underset{\substack{\uparrow \\ \text{One particular sol}^n\ \text{to } (+).}}{y_p}$$

EX: $y''' + y' = e^{-2t}$ $\quad (+)$

$y''' + y' = 0$ $\quad (*)$

Step 1: Solve $(*)$. Char. equ $r^3 + r = 0$

$r(r^2+1) = 0$

Roots: $r = 0, \pm i$

Get $y_1 = e^{0.t} \equiv 1$ sol^n.

Hmmm. e^{it} is a complex sol^n.

$$0 = L[e^{it}] = L[\cos t + i \sin t] = \underbrace{L[\cos t]}_{=0} + i\, \underbrace{L[\sin t]}_{=0}$$

Get two more real sol^{n}s: $y_2 = \cos t$, $y_3 = \sin t$.

Who cares how!

Is $W \neq 0$?

$$\det \begin{bmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{bmatrix}$$

$$= 1 \cdot \det \begin{bmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{bmatrix}$$

$$= \sin^2 t + \cos^2 t \equiv 1 \leftarrow \text{not zero!}$$

So $y = c_1 \cdot 1 + c_2 \cos t + c_3 \sin t$

is gen^l sol^n to homog eqn (*).

<u>Step</u>: Find <u>one</u> y_p. Try $y_p = Ae^{-2t}$

$$\left[Ae^{-2t}\right]''' + \left[Ae^{-2t}\right]' = e^{-2t} \quad \text{want}$$

$$-8Ae^{-2t} - 2Ae^{-2t} = e^{-2t} \leftarrow \text{divide out } e^{-2t}$$

$$-10A = 1 \qquad \boxed{A = -\tfrac{1}{10}}$$

Gen^l sol^n to (+): $y = \left(c_1 + c_2 \cos t + c_3 \sin t\right) - \frac{1}{10} e^{-2t}$