Lesson 20 Linear independence

Next HWK on home page due Wed

2nd order $y'' + Py' + Qy = 0$. Sol^{n}s y_1 and y_2

Good	Bad
1. $c_1y_1 + c_2y_2$ is Gen^l Sol^n	not
2. $W[y_1, y_2](x_0) \neq 0$	$W \equiv 0$
3. W is never zero	$\dfrac{W}{y_1^2} = \dfrac{y_1y_2' - y_2y_1'}{y_1^2} = \left(\dfrac{y_2}{y_1}\right)'$
1, 2, 3 are equivalent!	So $\left(\dfrac{y_2}{y_1}\right) = \text{const.}$
4. y_1 and y_2 are *linearly independent*.	One sol^n is a constant times the other: They are *linearly dependent*.

n-th order case: $y^{(n)} + P_{n-1}y^{(n-1)} + \cdots + P_1y' + P_0y = 0$

Get n sol^{n}s $y_1, y_2, \ldots, y_3$

Good	Bad
1. $c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is the Gen^l Sol^n	not
2. $W[y_1, \ldots, y_n](x_0) \neq 0$	

3. Abel's formula shows that

$W = Ce^{eee}$

So $W \equiv 0$ or never zero

4. $y_1, \cdots, y_n$ are linearly $\underline{\text{independent}}$ good

$y_1, \cdots, y_2$ are linearly $\underline{\text{dependent}}$ bad

$\underline{\text{Def}^n}$: $y_1, \cdots, y_n$ are linearly $\underline{\text{independent}}$ if

whenever (*) $c_1 y_1 + \cdots + c_n y_n = 0$, ← equal to the zero $\underline{\text{function}}$

$\underline{\underline{\text{all}}}$ the c's must be zero.

They are linearly $\underline{\text{dependent}}$ if this is not the case,

when there are $\underline{\text{not all zero } c\text{'s}}$ that make (*)

hold. Hmmm. Say $c_3 \neq 0$. ← Divide by c_3!

$$c_1 y_1 + c_2 y_2 + \underline{\underline{c_3}} y_3 + \cdots + c_n y_n \equiv 0$$

$$y_3 = -\frac{c_1}{c_3} y_1 - \cdots - \frac{c_n}{c_3} y_n$$ ← no y_3 term on right

Aha! One of the fcns is a linear combo of the others. ← Def^n of $\underline{\text{dependent}}$.

Fact: $W(x_0) \neq 0$, then sol'ns $y_1, \dots y_n$ are indep.

Why: Suppose $c_1 y_1 + \cdots + c_n y_n \equiv 0$

Diff'tiate:

$$\begin{cases} c_1 y_1 + \cdots + c_n y_n \equiv 0 \\ c_1 y_1' + \cdots + c_n y_n' \equiv 0 \\ \vdots \\ c_1 y_1^{(n-1)} + \cdots + c_n y_n^{(n-1)} \equiv 0 \end{cases}$$

$$\underbrace{\begin{bmatrix} \text{Wronskian} \\ \text{matrix} \end{bmatrix}}_{W = \det \neq 0} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \vec{0}$$

All c's must be zero! Indep!

Hmmm. What if $W(x_0) = 0$.

$$\underbrace{\begin{bmatrix} \text{Wronskian} \\ \text{matrix} \end{bmatrix}}_{\substack{\text{Bad matrix!} \\ \det = 0}} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \vec{0}$$

Row reduction:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} & 0 \\ & 0 \\ & 0 \\ & \\ 0 & 0 \end{bmatrix}$$

c_1 bound

c_2 bound

row of zeroes at bottom

Row of zeroes on bottom means not all var's are bound.

The non-bound are free variables. When free variables exist, there are not all zero c's sol^{n}s.

EX: If you have not all zero c's solving

$$\begin{bmatrix} \text{Wronskian} \\ \text{matrix} \end{bmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \vec{0}$$

$$\left\{ \begin{array}{l} c_1 y_1(x_0) + \cdots + c_n y_n(x_0) = 0 \\ \quad \vdots \\ c_1 y_1^{(n-1)}(x_0) + \cdots + c_n y_n^{(n-1)}(x_0) = 0 \end{array} \right\}$$

Aha! This linear combo satisfies the same IVP as the zero sol^n $y \equiv 0$.

Aha! The Uniqueness part of E & U Thm gives us that $\underbrace{c_1 y_1 + \cdots + c_n y_n}_{\text{not all } c\text{'s} = 0} \equiv 0$.

See $y_1, \ldots, y_n$ are dependent.

EX: Are $\cos x$ and $\sin x$ lin indep?

Duh! $\dfrac{\sin x}{\cos x} = \tan x \leftarrow$ not a constant!

EX: Are $\left\{ \begin{array}{l} 1 + 2x + 3x^2 \\ 4 + 5x + 6x^2 \\ 7 + 8x + 9x^2 \end{array} \right.$ indep?

No! $R3 = 2R2 - R1$

Scientific way:

$$c_1\left(1+2x+3x^2\right)+c_2\left(4+5x+6x^2\right)+c_3\left(7+8x+9x^2\right)\equiv 0$$

$$\underbrace{\left(c_1+4c_2+7c_3\right)}_{\text{must }=0}+\underbrace{\left(2c_1+5c_2+8c_3\right)}_{=0}x+\underbrace{\left(3c_1+6c_2+9c_3\right)}_{=0}x^2\equiv 0$$

$$\underbrace{\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}}_{\det=0!}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\vec{0}$$

There exist <u>not all zero c's</u> sol^{n}s.

[Same as (Row reduction) $\longrightarrow$ row zeroes on bottom.

Same as: free variables exist.

<u>Back to solving constant coeff linear homog eqns.</u>

$$a_n y^{(n)}+\cdots+a_1 y'+a_0 y=0$$

Try $y=e^{rx}$.

$$a_n r^n+a_{n-1}r^{n-1}+\cdots+a_1 r+a_0=0.$$

Fund Thm of Algebra: This n-th degree poly has n roots counted with multiplicity in $\mathbb{C}$.

Note: The a_n's are real. If a complex number z satisfies $P(z) = 0$, then

$$\underbrace{\overline{P(z)}}_{= P(\bar{z})} = 0$$

$$\overline{a_k z^k} = \bar{a}_k \overline{z^k} = \bar{a}_k \bar{z}^k$$

$\bar{a}_k = a_k$ when a_k real

and so we see that $\bar{z}$ is also a root.

Fact: Complex roots occur in conjugate pairs.

$$(r-z)(r-\bar{z}) = r^2 - \underbrace{(z+\bar{z})}_{2\,\mathrm{Re}\,z} r + \underbrace{z\bar{z}}_{|z|^2}$$ a real quadratic poly!

Big fact: Real polys factor into terms like this:

1) $(r - r_1)$ $\quad r_1$ real $\quad$ r_1 real root of multiplicity one

2) $(r - r_1)^m$ $\quad r_1$ real $\quad$ .... mult m

3) $(r^2 + br + c)$ $\quad$ roots $A \pm Bi$ conjugate pair

4) $(r^2 + br + c)^m$ $\quad$ roots $A \pm Bi$ mult. m