 n-th order linear ODE with constant coefficients   Hwk 8 due Wed

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = \begin{cases} 0 & \text{Homogeneous} \\ F(x) & \text{nonhomogeous} \end{cases}$$

Try $y = e^{rx}$ for homog eqn:

$$\underbrace{\left(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0\right)}_{\text{need } P(r)=0} e^{rt} = 0 \quad \leftarrow \text{want}$$

need $P(r) = 0$ ← need to factor.

$n=2$: quad formula
$n=3$: Cardano's
$n=4$: formula too
$n=5$ and above: Galois, Abel: No formula

Big fact: a_k's all real.

$P(r)$ factors into terms

like $(r-r_1)^{m_1}$ and $\underbrace{(r^2+br+c)^m}_{\text{Conjugate pair}} = (r-(A+Bi))^m (r-(A-Bi))^m$

r_1 real root of mult m

Conjugate pair of roots $A \pm Bi$ of mult. m.

EX: $P(r) = r^3 \underbrace{(r-2)}_{e^{2x}} (r-5)^4 \underbrace{(r^2+1)}_{r=\pm i} \underbrace{(r^2+2r+2)}^3 = 0$

r^3: $(r-0)^3$
$e^{0\cdot x}, xe^{0\cdot x}, x^2 e^{0\cdot x}$
$1, x, x^2$

$(r-5)^4$: $e^{5x}, xe^{5x}, x^2 e^{5x}, x^3 e^{5x}$

$r = \pm i$: $\cos x, \sin x$

$(r^2+2r+2)^3$: $r = -1 \pm i$

1. $e^{-x}\cos x$ 4. $e^{-x}\sin x$
2. $xe^{-x}\cos x$ 5. $xe^{-x}\sin x$
3. $x^2e^{-x}\cos x$ 6. $x^2 e^{-x}\sin x$

$P(r)$ of degree n: Get n linearly indep sol^{n}s $y_1, \ldots, y_n$.

Get gen^l sol^n to homog: $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

Cases: 1) $(r-r_1)$ simple factor. Get sol^n $e^{r_1 x}$.

2) $(r-r_1)^m$ with multiplicity.

$$P(r) = (r-r_1)^m \, Q(r) \leftarrow \quad Q(r_1) \neq 0$$

$$L[y] = a_n y^{(n)} + \cdots + a_0 y.$$

$$L[e^{rx}] = P(r)\, e^{rx}$$

Plug $r = r_1$. See $e^{r_1 x}$ is a sol^n

$$\frac{\partial}{\partial r} L[e^{rx}] = \frac{\partial}{\partial r}\left(P(r) e^{rx}\right) =$$

$$L\Big[\underbrace{\frac{\partial}{\partial r} e^{rx}}_{x e^{rx}}\Big] = \frac{\partial}{\partial r}\left[(r-r_1)^m Q(r)\, e^{rx}\right]$$

$$= m(r-r_1)^{m-1} Q e^{rx} + (r-r_1)^m \frac{\partial}{\partial r}\left[Q e^{rx}\right]$$

Euler: Aha! Let $r = r_1$

Find $x e^{r_1 x}$ as a 2nd sol^n.

Good news. Can repeat trick a total of $m-1$ times!

Get $x^2 e^{r_1 x}, x^3 e^{r_1 x}, \ldots, x^{m-1} e^{r_1 x}$

Case: $r^2 + br + c$ roots $A \pm Bi$.

For $A + Bi$: Get complex sol^n $e^{(A+Bi)x} = e^{Ax} e^{Bix}$

$$= e^{Ax}(\cos Bx + i \sin Bx)$$

$$= \underbrace{e^{Ax} \cos Bx} + i \, \underbrace{e^{Ax} \sin Bx}$$

two real sol^{n}s.

Case $(r^2+br+c)^m$ roots $A \pm Bi$

$$e^{(A \pm Bi)x} \quad \begin{cases} e^{Ax} \cos Bx \\ e^{Ax} \sin Bx \end{cases}$$

$$x e^{(A \pm Bi)x} \quad \begin{cases} x e^{Ax} \cos Bx \\ x e^{Ax} \sin Bx \end{cases}$$

$$\vdots$$

$$x^{m-1} e^{(A \pm Bi)x} \quad \begin{cases} x^{m-1} e^{Ax} \cos Bx \\ x^{m-1} e^{Ax} \sin Bx \end{cases}$$

Total of $2m$ lin indep sol'ns.

Fact: Guaranteed to be lin indep. (Wronskian $\neq 0$.)

EX: $y^{(4)} + 2y'' + y = 0$

$r^4 + 2r^2 + 1 = 0$

$(r^2+1)^2 = 0$

$r = \pm i$, mult 2 : $r = i, i, -i, -i$

Get $\begin{cases} \cos x \\ \sin x \end{cases}$ $\begin{cases} x \cos x \\ x \sin x \end{cases}$ 4 sol'ns.

Gen'l Sol'n $y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x$

Method of undet coeff: $\underbrace{a_n y^{(n)} + \cdots + a_0 y}_{\text{const coeff linear}} = \underbrace{F(x)}_{\text{special}}$

$F(x)$	Try $y_p =$
e^{ax}	$y_p = Ae^{ax}$
x^n or any poly of deg n	$A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$
$\cos \omega t$ $\sin \omega t$ or a linear combo	$y_p = A\cos\omega t + B\sin\omega t$
$\vdots$	
$x^n e^{ax}\cos bx$ or $x^n e^{ax}\sin bx$	$(A_n x^n + \cdots + A_0)e^{ax}\cos bx + (B_n x^n + \cdots + B_0)e^{ax}\sin bx$ e.g. $\to B_3 x^3 e^{ax}\sin bx$

Danger! If any single piece of the trial sol^n solves the homog eqn, method might fail. But in this case:

Rule: Get correct trial sol^n y_p by multiplying the trial sol^n from the table by x^m where m is the smallest positive that clears the danger.

EX: $y^{(4)} + 2y'' + y = x^2 \sin x$

Homog: $\cos x$, $\sin x$, $x\cos x$, $x\sin x$

<u>Table</u>: $(A_2x^2 + A_1x + A_0)\operatorname{Cos}x + (B_2x^2 + B_1x + B_0)\operatorname{Sin}x$

$$A_2x^2\operatorname{Cos}x + \underbrace{A_1x\operatorname{Cos}x} + \underbrace{A_0\operatorname{Cos}x} + B_2x^2\operatorname{Sin}x + B_1\underbrace{x\operatorname{Sin}x} + B_0\underbrace{\operatorname{Sin}x}$$

ouch!

Multiply by x: Ouch! still have $A_0x\operatorname{Cos}x$, $B_0x\operatorname{Sin}x$.

Correct trial solⁿ:

$$y_p = x^2\left[(A_2x^2 + \cdots + A_0)\operatorname{Cos}x + (B_2x^2 + \cdots + B_0)\operatorname{Sin}x\right]$$

<u>Finding roots of polys</u>:

$$r^5 - r^4 + r^3 - r^2 + r - 1 = 0$$

Aha! $r=1$ is a root! So $(r-1)$ divides Poly.

$$\begin{array}{r|l} & r^4 + r^2 + 1 \\ \hline r-1 & r^5 - r^4 + r^3 - r^2 + r - 1 \\ & r^5 - r^4 \\ \hline & \qquad\quad r^3 - r^2 + r - 1 \\ & \qquad\quad r^3 - r^2 \\ \hline & \qquad\qquad\quad r - 1 \\ & \qquad\qquad\quad r - 1 \\ \hline & \qquad\qquad\qquad 0 \end{array}$$

$$P(r) = (r-1)(r^4 + r^2 + 1)$$

<u>Famous observation</u>: Poly with rational coeff:

Clear denom from fractions. Get Poly with integer coeff.

How to look for rational roots $\pm \frac{p}{q}$:

$$a_n\left(\frac{p}{q}\right)^n + \cdots + a_0 = 0$$ Mult by q^n

$$\left(a_n p^n + \left(a_{n-1} p^{n-1} q + \cdots + a_1 p\right) + a_0 q^n\right) = 0$$

see q divides a_n see p divides a_0.

Get finite list of rational roots to test $\pm \frac{p}{q}$