Lesson 22

Variation of parameters for n-th order linear ODEs 4.4

new HWK on Home p. due Wed.

Fact: Hit or miss method to find roots to polynomials with rational coeff.

Step 1: Clear out all the fractions:

$\frac{2}{3}x^2 + \frac{1}{5}x + 1$ ← multiply by 15

$10x^2 + 3x + 15$

Integer coefficient poly: $a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0$

where a_n's are integers.

We want to find rational roots, if any. $r = \pm\frac{p}{q}$ ← no common factors

Plug in $r = \frac{p}{q}$ and clear fractions by multiplying by q^n:

$$\Big(a_n p^n + \big(a_{n-1} p^{n-1} q + \cdots + a_1 p q^{n-1} \big) + a_0 q^n \Big) = 0$$

↑ if $\frac{p}{q}$ root

any factor of q divides this

conclude that factors of q divides a_n

Green ( )'s show: any factor of p must divide a_0.

Red ( )'s show: any factor of q must divide a_n

Strategy: Find all factors of a_n: $\pm m_1, m_2, \ldots, m_k$

all factors of a_0: $\pm \ell_1, \ell_2, \ldots, \ell_q$

Check $P\left(\frac{p}{q}\right) = 0$ for $\frac{p}{q} = \pm\frac{\ell_j}{m_i}$.

Factors of 12: 1, 2, 3, 4, 6, 12

64: 1, 2, 4, 8, 16, 32, 64

Abel's formula for n-th order linear homog eqns:

$$y^{(n)} + \underline{\underline{p_1(x)}}\, y^{(n-1)} + \cdots + p_{n-1}(x)\, y' + p_n(x)\, y = 0$$

Suppose $y_1, y_2, \cdots, y_n$ solve (*).

Wronskian: $W = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & & \cdots & y_n^{(n-1)} \end{bmatrix}$.

Abel's: $\dfrac{dW}{dx} = -p_1(x)\, W$ So $W = Ce^{-\int p_1(x)\,dx}$

$C=0$: $W \equiv 0$

$C \neq 0$: W never 0

Fact: $\dfrac{d}{dx} \det \begin{bmatrix} \text{Row } 1 \\ \text{Row } 2 \\ \vdots \\ \text{Row } n \end{bmatrix} = \det \begin{bmatrix} (\text{Row } 1)' \\ \text{Row } 2 \\ \vdots \\ \text{Row } n \end{bmatrix} + \cdots + \det \begin{bmatrix} \text{Row } 1 \\ \text{Row } 2 \\ \vdots \\ (\text{Row } n)' \end{bmatrix}$

Why: $\dfrac{d}{dx} \det \begin{bmatrix} a & A \\ b & B \end{bmatrix} = (aB - Ab)' = a'B - A'b + aB' - Ab'$, etc.

Aha! $\dfrac{d}{dx}(\text{Row 1 of Wronskian matrix}) = \text{Row } 2 \leftarrow \det = 0$

$\dfrac{d}{dx}(\text{Row } 2) = \text{Row } 3 \qquad \leftarrow \det = 0$

$\vdots$

$\dfrac{d}{dx}(\text{Last row}) = \cdots$ ouch

$$\frac{d}{dx}W = \begin{bmatrix} y_1 & & y_n \\ y_1' & & y_2' \\ & & \\ y_1^{(n)} & & y_n^{(n)} \end{bmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} a\\ a \\ a\end{matrix}}\right\} \text{same as before} \\ \leftarrow \text{new bottom row}\end{array}$$

ODE

$$y_k^{(n)} = -p_1 y_k^{(n-1)} - \cdots - p_n y_k$$

$\uparrow$ Take transpose $\begin{pmatrix} y_1^{(n)} \\ \vdots \\ y_n^{(n)} \end{pmatrix}^T = (y_1^{(n)}, \dots, y_n^{(n)})$

Aha!

$$\frac{dW}{dx} = \det \begin{bmatrix} y_1 & & y_n \\ y_1' & & y_n' \\ & & \\ -p_1 y_1^{(n-1)} & & -p_1 y_n^{(n-1)} \end{bmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} a\\ a \\ a\end{matrix}}\right\} \text{same as before} \\ \leftarrow \text{new}\end{array}$$

$$= -p_1 \det[\quad] \checkmark$$

Method of Variation of Paramters for n-th order linear ODEs

$$y^{(n)} + p_1(x) y^{(n-1)} + \cdots + p_n(x) y = F(x)$$

Step 1: Get gen^l sol^n to homog:

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \quad \leftarrow c\text{'s parameters}$$

Step 2: Try $y_p = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n \leftarrow u$'s are fcns

Brilliant idea: Need $\underline{\underline{n}}$ conditions to pin down n u's!

$$y_p' = \underbrace{\left(u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n\right)}_{\text{make this } = 0} + \left(u_1 y_1' + \cdots + u_n y_n'\right)$$

$$y_p'' = \underbrace{\left(u_1' y_1' + \cdots + u_n' y_n'\right)}_{\text{make this } = 0} + \left(u_1 y_1'' + \cdots + u_n y_n''\right)$$

$$\vdots$$

$$y_p^{(n)} = \left(u_1' y_1^{(n-1)} + \cdots + u_n' y_1^{(n-1)}\right) + \left(u_1 y_1^{(n)} + \cdots + u_n y_n^{(n)}\right)$$

Read 4.4. Get

$$\begin{pmatrix} u_1' y_1 + \cdots + u_n' y_n \\ u_1' y_1' + \cdots + u_n' y_n \\ \vdots \qquad\qquad \vdots \\ u_1' y_1^{(n-1)} + \cdots + u_n' y^{(n-1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ F(x) \end{pmatrix}$$

$$\begin{bmatrix} \text{Wronskian} \\ \text{matrix} \end{bmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}' = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ F(x) \end{pmatrix}$$

Use Cramer's rule to get u' s. k-th column

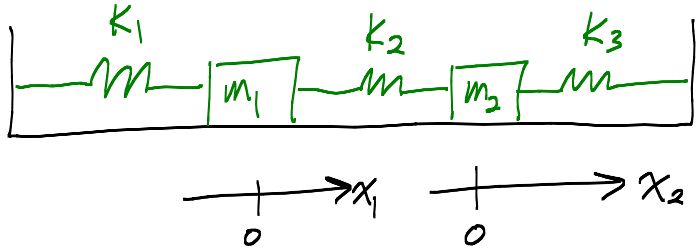
$$u_k' = \frac{\det \begin{bmatrix} & \begin{matrix} 0 \\ \vdots \\ 0 \\ F \end{matrix} & \end{bmatrix}}{\det \begin{bmatrix} \text{Wronskian} \\ \text{matrix} \end{bmatrix}} = \frac{F \det [\quad]}{W}$$

k-th

How bad is this really? Abel's formula to the rescue!

$$W = Ce^{-\int p_1\, dx}$$ ← plug in $x=0$ to determine C.

Skip ahead to Chap 7. Systems



Slack position

Net force on m_1: $F = ma$

$$\begin{cases} -k_1 x_1 + k_2(x_2 - x_1) = m_1 \ddot{x}_1 \\ -k_3 x_2 - k_2(x_2 - x_1) = m_2 \ddot{x}_2 \end{cases}$$