Lesson 23 Var of Par, Systems (Chap. 7). HWK 9 due Wed

Method of Variation of Paramters for n-th order linear ODEs

$$L[y] = 1 \cdot y^{(n)} + p_1(x)\, y^{(n-1)} + \cdots + p_n(x)\, y = F(x)$$ ← Standard Form

Step 1: Get gen'l sol'n to homog:

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

c's parameters
u's are fcns

Step 2: Try $y_p = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n$ $\quad * \; p_n(x)$

$$y_p' = \underbrace{\left(u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n\right)}_{\text{make this } = 0} + \left(u_1 y_1' + \cdots + u_n y_n'\right) \quad * \; p_{n-1}(x)$$

$$y_p'' = \underbrace{\left(u_1' y_1' + \cdots + u_n' y_n'\right)}_{\text{make this } = 0} + \left(u_1 y_1'' + \cdots + u_n y_n''\right) \quad * \; p_{n-2}(x)$$

$\vdots$

$$y_p^{(n-1)} = \underbrace{\left(u_1' y_1^{(n-2)} + \cdots + u_n' y_n^{(n-2)}\right)}_{\text{make this } = 0} + \left(u_1 y_1^{(n-1)} + \cdots + u_n y_n^{(n-1)}\right) \quad * \; p_1(x)$$

$$y_p^{(n)} = \left(u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)}\right) + \left(u_1 y_1^{(n)} + \cdots + u_n y_n^{(n)}\right) \quad * \; 1$$

$$+ \left(u_1 \underbrace{L[y_1]}_{0} + u_2 \underbrace{L[y_2]}_{0} + \cdots + u_n \underbrace{L[y_n]}_{0}\right) + \underbrace{\left(u_1' y_1^{(n-1)} + \cdots u_n' y_n^{(n-1)}\right)}_{\text{the last eqn for u's !}} = F(x)$$

want

$$\begin{pmatrix} u_1' y_1 + \cdots + u_n' y_n \\ u_1' y_1' + \cdots + u_n' y_n \\ \vdots \qquad\qquad \vdots \\ u_1' y_1^{(n-1)} + \cdots + u_n' y^{(n-1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ F(x) \end{pmatrix}$$

$$\begin{bmatrix} \text{Wronskian} \\ \text{matrix} \end{bmatrix} \begin{pmatrix} u_1' \\ \vdots \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ F(x) \end{pmatrix}$$

Use Cramer's rule to get u' s.

k-th column

$$u_k' = \frac{\det \begin{bmatrix} & 0 & \\ & \vdots & \\ & 0 & \\ & F & \end{bmatrix}}{\det \begin{bmatrix} \text{Wronskian} \\ \text{matrix} \end{bmatrix}} = \frac{F \det \begin{bmatrix} & 0 & \\ & \vdots & \\ & 0 & \\ \hline \end{bmatrix}}{W}$$

k-th

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \end{bmatrix}$$

EX: $x^3 y''' + x^2 y'' - 2xy' + 2y = 6x^4$

Step 1: Solve homog. Euler equ. Try $y = x^r$

Get $y_1 = x$, $y_2 = x^2$, $y_3 = x^{-1}$

Step 2: Need Standard Form to get $F(x)$.

$$y''' + \frac{1}{x} y'' - \frac{2}{x^2} y' + \frac{2}{x^3} y = 6x \quad \leftarrow F(x)$$

Step 3: Use formulas.

$$u_1' = \frac{\det\begin{bmatrix} 0 & x^2 & x^{-1} \\ 0 & 2x & -x^{-2} \\ 6x & 2 & 2x^{-3} \end{bmatrix}}{\det\begin{bmatrix} x & x^2 & x^{-1} \\ 1 & 2x & -x^{-2} \\ 0 & 2 & 2x^{-3} \end{bmatrix}}$$

$$= (6x)(+1)\det\begin{bmatrix} x^2 & x^{-1} \\ 2x & -x^{-2} \end{bmatrix} = -18x$$

$$\det\begin{bmatrix} x & x^2 & x^{-1} \\ 1 & 2x & -x^{-2} \\ 0 & 2 & 2x^{-3} \end{bmatrix} = -\det\begin{bmatrix} 1 & 2x & -x^{-2} \\ x & x^2 & x^{-1} \\ 0 & 2 & 2x^{-3} \end{bmatrix}$$

$$= -\det\begin{bmatrix} 1 & 2x & -x^{-2} \\ 0 & -x^2 & x^{-1} \\ 0 & 2 & 2x^{-3} \end{bmatrix} \quad R2+(-x)R1$$

$$= -(1)\det\begin{bmatrix} -x^2 & x^{-1} \\ 2 & 2x^{-3} \end{bmatrix} = \underbrace{6x^{-1}}_{W}$$

So $u_1' = \dfrac{-18x}{6x^{-1}} = -3x^2$

and $\boxed{u_1 = \int -3x^2\,dx = -x^3}$ ← No $+C$

$$u_2' = \frac{\det\begin{bmatrix} x & 0 & x^{-1} \\ 1 & 0 & -x^{-2} \\ 0 & 6x & 2x^{-3} \end{bmatrix}}{W} = \frac{(6x)(-1)\det\begin{bmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{bmatrix}}{6x^{-1}} = 2x$$

So $\boxed{u_2 = \int 2x\,dx = x^2}$

$$u_3' = \frac{\det\begin{bmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 6x \end{bmatrix}}{W} = \frac{6x(+1)\det\begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix}}{6x^{-1}} = x^4$$

So $\boxed{u_3 = \int x^4\,dx = \frac{1}{5}x^5}$

Finally: $y_p = u_1y_1 + u_2y_2 + u_3y_3 = (-x^3)\cdot x + (x^2)\cdot x^2 + \left(\frac{1}{5}x^5\right)x^{-1}$

$$= \frac{1}{5}x^4$$

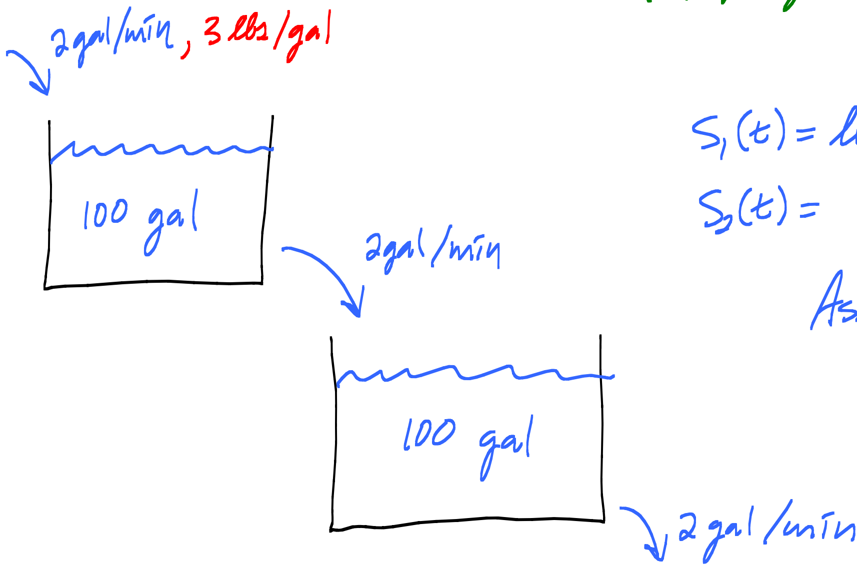
Gen^l Sol^n to non-homog:

$$\left(c_1 x + c_2 x^2 + c_3 x^{-1}\right) + \left(\frac{1}{5}x^4\right)$$

Great simplifying trick: $W = Ce^{-\int p_1(x)\,dx}$ $\quad p_1(x) = \frac{1}{x}$ here.

find C by plugging in $x = 1$.

Systems:



$S_1(t)$ = lbs salt in tank 1

$S_2(t)$ = - - - - - - - 2

Assume $\begin{cases} S_1(0) = 0 \\ S_2(0) = 0 \end{cases}$

$$\begin{cases} \dfrac{dS_1}{dt} = (\text{Rate in}) - (\text{Rate out}) \\ \quad = (2\,\text{gal/min})(3\,\text{lbs/gal}) - (2\,\text{gal/min})\left(\dfrac{S_1(t)}{100\,\text{gal}}\right) \\ \dfrac{dS_2}{dt} = (\text{Rate in}) - (\text{Rate out}) \\ \quad = (2\,\text{gal/min})\left(\dfrac{S_1}{100}\right) - (2\,\text{gal/min})\left(\dfrac{S_2(t)}{100\,\text{gal}}\right) \end{cases}$$

$$\begin{cases} \dfrac{dS_1}{dt} = -\dfrac{1}{50}S_1 + \underline{\underline{6}} \\ \dfrac{dS_2}{dt} = \dfrac{1}{50}S_1 - \dfrac{1}{50}S_2 \end{cases}$$

2x2 First order system.

non-homog

High school method:

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 - 3x_2 & (A) \\[2ex] \dfrac{dx_2}{dt} = -2x_1 + 2x_2 & (B) \end{cases}$$

Solve (A) for x_2: $\quad x_2 = -\frac{1}{3}\dot{x}_1 + \frac{1}{3}x_1$

So $\quad \dot{x}_2 = \frac{d}{dt}\left[-\frac{1}{3}\dot{x}_1 + \frac{1}{3}x_1\right] = -\frac{1}{3}\ddot{x}_1 + \frac{1}{3}\dot{x}_1$

Now plug into (B):

$$\underbrace{\left[-\frac{1}{3}\ddot{x}_1 + \frac{1}{3}\dot{x}_1\right]}_{\dot{x}_2} = -2x_1 + 2\underbrace{\left[-\frac{1}{3}\dot{x}_1 + \frac{1}{3}x_1\right]}_{x_2}$$

$$\boxed{\ddot{x}_1 - 3\dot{x}_1 - 4x_1 = 0}$$

$$r^2 - 3r - 4 = 0$$

$$(r-4)(r+1) = 0 \qquad r = -1, 4$$

So $\quad \boxed{x_1 = c_1 e^{-t} + c_2 e^{4t}}$

Now $\quad x_2 = -\frac{1}{3}\dot{x}_1 + \frac{1}{3}x_1 = -\frac{1}{3}\left(-c_1 e^{-t} + 4c_2 e^{4t}\right) + \frac{1}{3}\left(c_1 e^{-t} + c_2 e^{4t}\right)$

$$\boxed{x_2 = \frac{2}{3}c_1 e^{-t} - c_2 e^{4t}}$$

Two c's. Need 2 initial conditions to pin them down: $\begin{cases} x_1(0) = A \\ x_2(0) = B \end{cases}$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} \quad \longleftarrow \text{sol}^{n}$$

eigenvectors for $\mathbb{A}$!

$r = -1, 4$ are eigenvalues!

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 - 3x_2 \\ -2x_1 + 2x_2 \end{pmatrix} = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\vec{x}\,' = \mathbb{A}\vec{x}$ where $\mathbb{A} =$ matrix