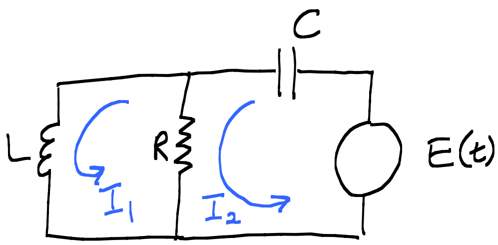
Lesson 24 Systems 7.1, 7.2 HWK 9 due Wed.



$$R(\dot{I}_2-\dot{I}_1)+\frac{1}{C}I_2=\dot{E}(t)$$

$$L\ddot{I}_1+R(\dot{I}_1-\dot{I}_2)=0$$

Remark: EX: $y''+5y'+y^2=\cos t$ (*)

Can turn a higher order ODE into a first order system

Trick: Let $x_1=y(t)$

$x_2=y'(t)$

First eqn easy: $\dfrac{dx_1}{dt}=y'(t)=x_2$

$$\frac{dx_2}{dt}=y''(t)\underset{\substack{\uparrow\\ \text{from (*)}}}{=}-5y'-y^2+\cos t$$

$$=-5x_2-x_1^2+\cos t$$

$$\begin{cases}\dfrac{dx_1}{dt}=x_2\\[2ex] \dfrac{dx_2}{dt}=-x_1^2-5x_2+\cos t\end{cases}$$

← ode45 makes you do this in matlab

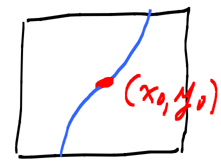
2x2 Systems:

$$\begin{cases}\dfrac{dx}{dt}=f(x,y,t)\\[2ex] \dfrac{dy}{dt}=g(x,y,t)\end{cases} \qquad \begin{cases}x(t_0)=x_0\\ y(t_0)=y_0\end{cases}$$

1st order 2x2 system.

Initial conditions

E & U Thm If f, g, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial y}$ are all continuous on a rectangle, then sol^{n}s to the IVP exist on a time interval about t_0 (that might be short), and sol^n is unique.



Linear systems:

2x2

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}(t)\,x_1 + a_{12}(t)\,x_2 + g_1(t) \\[2ex] \dfrac{dx_2}{dt} = a_{21}(t)\,x_1 + a_{22}(t)\,x_2 + g_2(t) \end{cases}$$

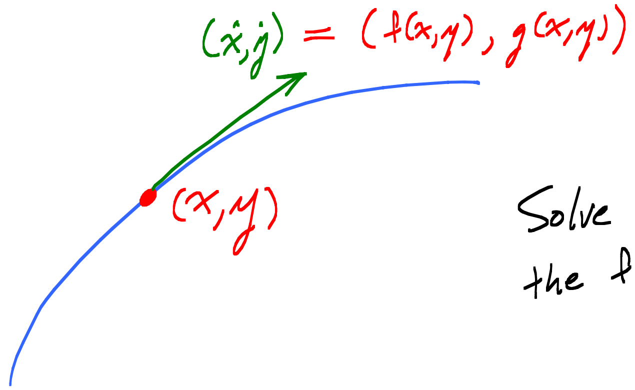
I.C. $\begin{cases} x_1(t_0) = A \\ x_2(t_0) = B \end{cases}$

E & U Thm If all the a_{ij}'s and g_j's are continuous near t_0, then sol^{n}s to the IVP exist and are unique on largest interval about t_0 where all the a's and g's are continuous.

Special case: Autonomous 2x2 systems:

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\[2ex] \dfrac{dy}{dt} = g(x,y) \end{cases} \quad \longleftarrow \text{no } t$$

Think $(x(t), y(t))$ curve in $\mathbb{R}^2$ in parametric form.



Solve system: Find the flow along a field.

Linear constant coeff homogeneous 1st order systems

$$\begin{cases} \dfrac{dx_1}{dt} = 5x_1 - x_2 & (A) \\[2ex] \dfrac{dx_2}{dt} = 3x_1 + x_2 & (B) \end{cases}$$

Last time: High school method: Solve (A) for x_2. Plug into (B). Get 2^{nd} order ODE in x_1. Solve it. Now get x_2 from (A).

Got
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} A \\ B \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} C \\ D \end{pmatrix} e^{r_2 t}$$

$$\vec{x} = c_1 \vec{a}_1 e^{r_1 t} + c_2 \vec{a}_2 e^{r_2 t}$$

College method: Look for sol^{n}s of form $\vec{x} = \vec{a} e^{rt}$.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{rt} = \begin{pmatrix} a_1 e^{rt} \\ a_2 e^{rt} \end{pmatrix} \qquad \vec{x} = \vec{a} e^{rt}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} r a_1 e^{rt} \\ r a_2 e^{rt} \end{pmatrix} \qquad \vec{x}\,' = r \vec{a} e^{rt}$$

Plug in and force it.

$$\begin{cases} \dot{x}_1 = 5x_1 - x_2 \\ \dot{x}_2 = 3x_1 + x_2 \end{cases}$$

$$\vec{x}\,' = \underbrace{\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}}_{\mathbb{A}} \vec{x}$$

$$\begin{cases} r a_1 e^{rt} = 5a_1 e^{rt} - a_2 e^{rt} \\ r a_2 e^{rt} = 3a_1 e^{rt} + a_2 e^{rt} \end{cases}$$

$$r \vec{a} e^{rt} = \mathbb{A} \vec{a} e^{rt}$$

Cancel out e^{rt}'s

$$\boxed{r \vec{a} = \mathbb{A} \vec{a}}$$

r eigenvalue for eigenvector $\vec{a}$

$$\begin{cases} 0 = (5-r) a_1 - a_2 \\ 0 = 3 a_1 + (1-r) a_2 \end{cases}$$

$$\boxed{(\mathbb{A} - r\mathbb{I}) \vec{a} = \vec{0}}$$

$$\begin{bmatrix} 5-r & -1 \\ 3 & 1-r \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Think: Cramer's rule:

$$a_1 = \frac{\det \begin{bmatrix} 0 & -1 \\ 0 & 1-r \end{bmatrix}}{\det \begin{bmatrix} 5-r & -1 \\ 3 & 1-r \end{bmatrix}} = 0 \,?$$

Want non-zero $\vec{a}$ sol^{n}s! Want Cramer's to fail!

Need

$$\boxed{\det(\mathbb{A} - r\mathbb{I}) = 0}$$

$$\det \begin{bmatrix} 5-r & -1 \\ 3 & 1-r \end{bmatrix} = (5-r)(1-r) + 3 = 0$$

$$\boxed{r^2 - 6r + 8 = 0}$$

$$(r-2)(r-4) = 0$$

Get two eigenvalues for A: $r = 2, 4$

Next, find e.vectors:

For $r=2$: $(A - 2I)\vec{a} = \vec{0}$

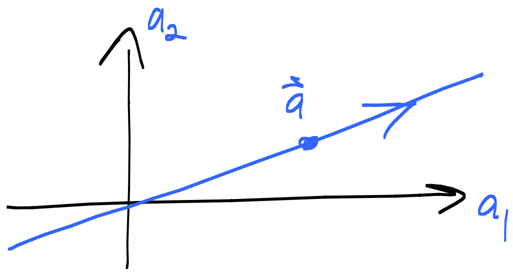
$$\begin{bmatrix} 5-2 & -1 \\ 3 & 1-2 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad 3a_1 - a_2 = 0$$

Think: a_1 is a bound variable. a_2 is free.

Let $a_2 = t$. Then $a_1 = \frac{1}{3} a_2 = \frac{1}{3} t$



Just want one non-zero sol^n.

Take $t=3$ to clear fraction.

Get nice $\vec{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Got $\begin{pmatrix}1\\3\end{pmatrix}e^{2t}$!

For $r=4$:
$$\left[\begin{array}{cc|c} 5-4 & -1 & 0 \\ 3 & -1-4 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right] \leftarrow \text{2nd eqn} = 3(\text{first})$$

$a_1 - a_2 = 0$

a_1 bound. a_2 free. Let $a_2 = t$.

$a_1 = a_2 = t$ $\quad \vec{a} = \begin{pmatrix}t\\t\end{pmatrix} = t\begin{pmatrix}1\\1\end{pmatrix}$. Take $t=1$.

Get $\begin{pmatrix}1\\1\end{pmatrix}e^{4t}$

Superposition principle holds for linear homg syst.

Get sol'ns $\quad \vec{x} = c_1\begin{pmatrix}1\\3\end{pmatrix}e^{2t} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{4t}$

Is this the gen'l sol'n? Equivalently: Can I solve any IVP with these?

Take $t_0 = 0$: $\quad \vec{x}(0) = c_1\begin{pmatrix}1\\3\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}A\\B\end{pmatrix}$

$$\underbrace{\begin{bmatrix}1 & 1\\ 3 & 1\end{bmatrix}}_{\text{need } \det \neq 0.\ \checkmark}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}A\\B\end{pmatrix}$$

Can get c's for any RHS. E & U Thm implies I

get the unique sol^n!