Lesson 25 on 2x2 linear homog systems with constant coeff

$$\begin{cases} \dfrac{dx_1}{dt} = 5x_1 - x_2 \\ \dfrac{dx_2}{dt} = 3x_1 + x_2 \end{cases}$$

$$\vec{x}\,' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \vec{x} \qquad \vec{x}\,' = \mathbb{A}\vec{x}$$

$$\begin{cases} x_1 = c_1 e^{2t} + c_2 e^{4t} \\ x_2 = 3c_1 e^{2t} + c_2 e^{4t} \end{cases} \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\underline{\vec{x} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}}$$

Ques: Is this the gen^l sol^n? Can I solve all IVP's with it?

$$\begin{cases} x_1(t_0) = A \\ x_2(t_0) = B \end{cases}$$

$$c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t_0} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t_0} = \begin{pmatrix} A \\ B \end{pmatrix}$$

$$c_1 \begin{pmatrix} e^{2t_0} \\ 3e^{2t_0} \end{pmatrix} + c_2 \begin{pmatrix} e^{4t_0} \\ e^{4t_0} \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\underbrace{\begin{bmatrix} e^{2t_0} & e^{4t_0} \\ 3e^{2t_0} & e^{4t_0} \end{bmatrix}}_{\text{need } \det \neq 0} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

could be anything

$$\det = e^{2t_0} e^{4t_0} - 3e^{2t_0} e^{4t_0} = -2 e^{6t_0} \leftarrow \text{never zero!}$$

Yes! This yields a gen^l sol^n for IVPs at t_0.

Def^n: Two sol^{n}s $\vec{x}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}$ $\quad \vec{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

Let $\mathbb{X} = [\vec{x}_1, \vec{x}_2]$, $\det \mathbb{X} \neq 0$.

$\mathbb{X}$ is called a <u>Fundamental Matrix</u> for $\vec{x}' = \mathbb{A}\vec{x}$.

The Wronskian is $\det \mathbb{X}$. $\left[W = W(\vec{x}_1, \vec{x}_2) = \det \mathbb{X} \right]$

Fancy notation: $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2 = [\vec{x}_1, \vec{x}_2] \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbb{X} \vec{c}$

<u>Wronskian fact</u>: Abel's formula shows W is always zero or never zero.

<u>EX</u>: $\begin{cases} \dfrac{dx_1}{dt} = x_1 - 3x_2 \\ \dfrac{dx_2}{dt} = -2x_1 + 2x_2 \end{cases}$

$\vec{x}' = \mathbb{A}\vec{x}$ where

$$\mathbb{A} = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

Try sol^{n}s of form $\vec{x} = \vec{a} e^{rt}$.

r = eigenvalue for $\mathbb{A}$

$\vec{a}$ eigenvector for r.

How to find r's: $\boxed{\det(\mathbb{A} - r\mathbb{I}) = 0}$

$$\det \begin{bmatrix} 1-r & -3 \\ -2 & 2-r \end{bmatrix} = 0$$

$$(2-r)(1-r) - 6 = 0$$

$$r^2 - 3r - 4 = 0$$

$$(r-4)(r+1) = 0$$

$\boxed{r = -1, 4}$ e. vals

For $r=4$:

$$\boxed{(\mathbb{A}-r\mathbb{I})\vec{a}=\vec{0}}$$ ← e.vect sys for e.val r

$$\left[\begin{array}{cc|c} 1-4 & -3 & 0 \\ -2 & 2-4 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -3 & -3 & 0 \\ -2 & -2 & 0 \end{array}\right]$$ ← Eqn 2 = $\frac{2}{3}$ Eqn 1

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$ ← $a_1+a_2=0$

a_1 bound var

a_2 free

Let $a_2=t$ (anything). Then $a_1=-a_2=-t$

Get $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix}$. Just want one non-zero $\vec{a}$.

How about $t=-1$. Get $\vec{a}=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Get $\boxed{\vec{x}_1=\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{4t}}$

For $r=-1$: $(\mathbb{A}-(-1)\mathbb{I})\vec{a}=\vec{0}$

$$\left[\begin{array}{cc|c} 1-(-1) & -3 & 0 \\ -2 & 2-(-1) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & -3 & 0 \\ -2 & 3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad 2a_1 - 3a_2 = 0$$

$\uparrow$ a_1 bound $\qquad$ $\uparrow$ a_2 free

$$a_1 = \frac{3}{2} a_2$$

$a_2 = t$. $\quad a_1 = \frac{3}{2} a_2 = \frac{3}{2} t \qquad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} t \\ t \end{pmatrix}$

Take $t = 2$: $\quad$ Get $\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. $\quad \boxed{\vec{x}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t}}$

Superposition princ: $\quad \vec{x}_1{}' = \mathbb{A} \vec{x}_1$

$$\vec{x}_2{}' = \mathbb{A} \vec{x}_2$$

$$(c_1 \vec{x}_1 + c_2 \vec{x}_2)' = c_1 \vec{x}_1{}' + c_2 \vec{x}_2{}' = c_1 \mathbb{A} \vec{x}_1 + c_2 \mathbb{A} \vec{x}_2$$

$$= \mathbb{A} (c_1 \vec{x}_1 + c_2 \vec{x}_2)$$

a sol^n!

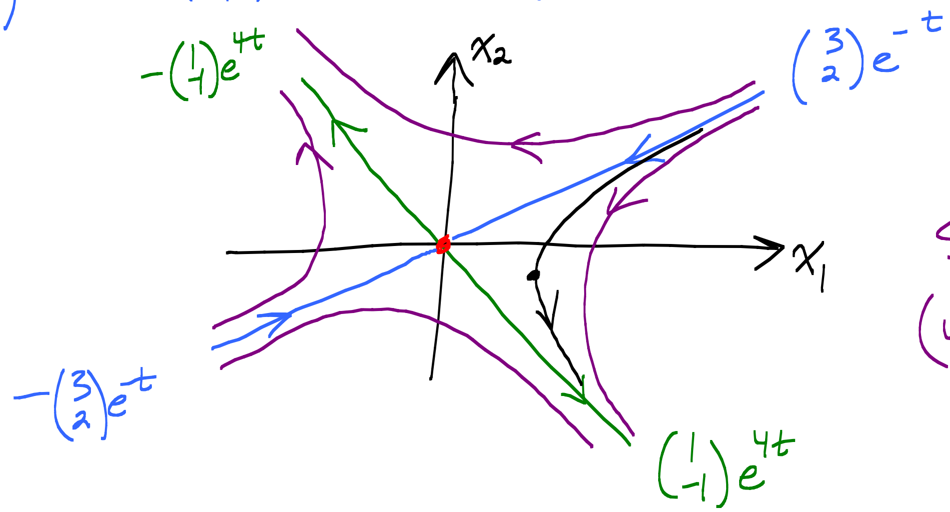
$$\mathbb{X} = \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t} \right] = \begin{bmatrix} e^{4t} & 3e^{-t} \\ -e^{4t} & 2e^{-t} \end{bmatrix}$$

$$W = \det \mathbb{X} = 2e^{3t} + 3e^{3t} = 5e^{3t} \neq 0$$

Fact: Method always spits out a gen^l sol^n if $r_1 \neq r_2$.

Fun to graph sol^{n}s in "phase plane"

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t}$$



Saddle point (unstable)

Complex evals:

$$\begin{cases} \dfrac{dx_1}{dt} = -2x_1 + 2x_2 \\[2ex] \dfrac{dx_2}{dt} = -2x_1 - 2x_2 \end{cases} \qquad \mathbb{A} = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}$$

$$\det \begin{bmatrix} -2-r & 2 \\ -2 & -2-r \end{bmatrix} = (-2-r)^2 + 4 = r^2 + 4r + 8 = 0$$

$$r = \frac{-4 \pm \sqrt{16-32}}{2} = \frac{-4 \pm 4i}{2} = \underline{\underline{-2 \pm 2i}}$$

Pretend not to notice!

For $r = -2+2i$:

$$\left[\begin{array}{cc|c} -2-(-2+2i) & 2 & 0 \\ -2 & -2-(-2+2i) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right] = -i \cdot \text{Eqn } 1$$

$$\rightsquigarrow \left[\begin{array}{cc|c} -2i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{cc} \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right] & a_1 + i a_2 = 0 \\ \begin{array}{cc} a_1 & a_2 \\ \text{bound} & \text{free} \end{array} & \end{array}$$

Let $a_2 = t$. $\quad a_1 = -i a_2 = -i t$

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -it \\ t \end{pmatrix}. \qquad \text{Take } t = i.$$

Get $\vec{a} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

Got a complex sol^n $\vec{x} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-2+2i)t}$

$$= \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \left(e^{-2t} \cos 2t + i e^{-2t} \sin 2t\right)$$

$$= \left[\underline{\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} \cos 2t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t} \sin 2t}\right] + i \left[\underline{\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} \sin 2t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t} \cos 2t}\right]$$

$\vec{x}_1$ $\vec{x}_2$

Get two real sol^{n}s! $W \neq 0$.

Get gen^{l} sol^{n}

$$\vec{x} = c_1 \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^{-2t}$$