Lesson 26 Linear first order homogeneous 2x2 systems with constant coefficients

General 1st order homog sys:
$$\begin{cases} \dfrac{dx_1}{dt} = p_{11}(t)x_1 + p_{12}(t)x_2 \\ \dfrac{dx_2}{dt} = p_{21}(t)x_1 + p_{22}(t)x_2 \end{cases}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad W(\vec{x}_1, \vec{x}_2) = \det[\vec{x}_1, \vec{x}_2]$$

Abel's formula: $W = Ce^{\int p_{11}(t) + p_{22}(t)\,dt}$

Last time:
$$\begin{cases} \dfrac{dx_1}{dt} = -2x_1 + 2x_2 \\ \dfrac{dx_2}{dt} = -2x_1 - 2x_2 \end{cases}$$

$$\vec{x} = c_1 \underbrace{\begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^{-2t}}_{\vec{x}_1} + c_2 \underbrace{\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^{-2t}}_{\vec{x}_2}$$

$$W = Ce^{\int (-2-2)\,dt} = Ce^{-4t}$$

$$W(0) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$
$$\begin{array}{cc} \uparrow & \uparrow \\ \vec{x}_1(0) & \vec{x}_2(0) \end{array}$$

Aha! $W(0) = C = 1$. So $W = e^{-4t}$.

Remark: This $\vec{x}_1$ and $\vec{x}_2$ is a great fundamental set of sol'ns because the fundamental matrix $X = [\vec{x}_1, \vec{x}_2]$ is such that
$$X(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Makes it very easy to solve I.V.P.s: $\vec{x}(0) = \begin{pmatrix} A \\ B \end{pmatrix}$

$$\vec{x}(0) = c_1 \vec{x}_1(0) + c_2 \vec{x}_2(0) \overset{\text{want}}{=} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \quad \Leftarrow \begin{array}{l} c_1 = A \\ c_2 = B \end{array} \quad \text{easy!}$$

When you have $X(0) = I$, call it a <u>normalized</u> fund matrix.

Call it Φ.

<u>Graph</u>: $c_1 \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^{-2t}$ in phase plane.

Sine, cosine stuff is π-periodic. e^{-2t} is shrinking.

This is a <u>spiral</u> in to the origin.

[Cross out e^{-2t}'s, these are ellipses.]

<u>Question</u>: Are spirals clockwise or counterclockwise?

$(0,0)$ is a <u>critical point</u>: $\begin{cases} x_1(t) \equiv 0 \\ x_2(t) \equiv 0 \end{cases}$ solves system.

If I "poke" a sol^n away from $(0,0)$, it spirals back in.

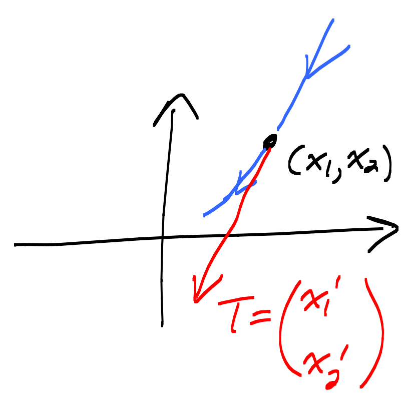
$(0,0)$ is <u>asymptotically</u> <u>stable</u>.

CW or CCW? Test field along positive real axis.

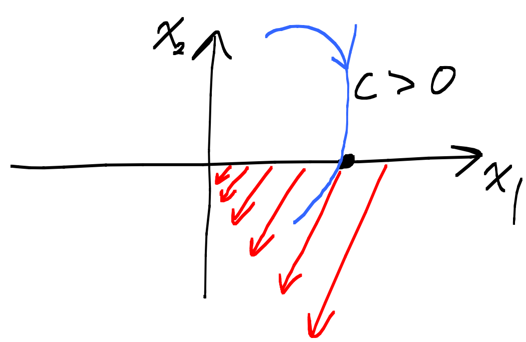
$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Tangent vector

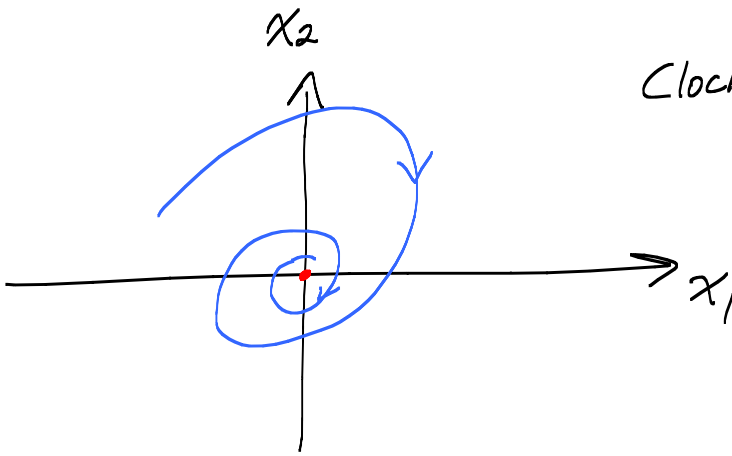
pt. on curve



Clockwise!



$$\vec{x}' = \begin{pmatrix} -2 & 2 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} -2c \\ -2c \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} c$$



Clockwise spiral <u>in</u>.

<u>Fact</u>: Complex sol^n's give rise to <u>two</u> real sol^n's. [$\mathbb{A}$ real matrix]

$$\vec{\mathbb{Y}} = \vec{u} + i\vec{v}$$

$$\vec{\mathbb{Y}}' = \mathbb{A}\vec{\mathbb{Y}} \leftarrow \text{one complex sol}^n$$

$$(\vec{u}+i\vec{v})' = \mathbb{A}(\vec{u}+i\vec{v}) = \mathbb{A}\vec{u} + i\mathbb{A}\vec{v}$$

$$\underbrace{(\vec{u}' - \mathbb{A}\vec{u})}_{\text{must be } \vec{0}} + i\underbrace{(\vec{v}' - \mathbb{A}\vec{v})}_{\text{must be } \vec{0}} = \vec{0}$$

$\vec{u}, \vec{v}$ <u>two</u> real sol^n's.

<u>EX</u>: Important skill:

Complex sol^n $\rightarrow$

$$\begin{pmatrix} 3+5i \\ 4+6i \end{pmatrix} e^{(2+3i)t} = \left[\begin{pmatrix} 3 \\ 4 \end{pmatrix} + i\begin{pmatrix} 5 \\ 6 \end{pmatrix}\right]\left(e^{2t}\cos 3t + i\,e^{2t}\sin 3t\right)$$

$$= \underbrace{\left[\begin{pmatrix} 3 \\ 4 \end{pmatrix} e^{2t}\cos 3t \underset{\uparrow\, i^2=-1}{-} \begin{pmatrix} 5 \\ 6 \end{pmatrix} e^{2t}\sin 3t\right]}_{\vec{x}_1} + i\underbrace{\left[\begin{pmatrix} 3 \\ 4 \end{pmatrix} e^{2t}\sin 3t + \begin{pmatrix} 5 \\ 6 \end{pmatrix} e^{2t}\cos 3t\right]}_{\vec{x}_2}$$

$\vec{x}_1, \vec{x}_2$ are real sol^{n}s. ($W \neq 0$ guaranteed.)

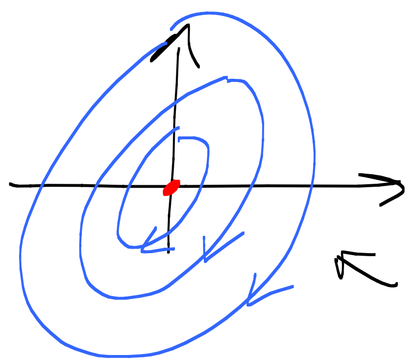
Fact: Complex eigenvalues $r = a \pm bi$

Get $(\text{vectors})\, \underline{\underline{e^{at}}}\, (\text{Sine and Cosine } bt)$

$a<0$ spiral in

$a>0$ spiral out

a=0 case:



A "center"

ellipses

(0,0) Stable, but not asymptotically stable.

sol^{n}s stay close by under small "poke"

sol^{n}s return to (0,0)

EX: $\vec{x}\,' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \vec{x}$

$$\det(A - rI) = \det \begin{bmatrix} 5-r & -1 \\ 3 & 1-r \end{bmatrix} = (5-r)(1-r)+3 = r^2-6r+8$$

$$= (r-2)(r-4) = 0$$

e.vals: $r = 2, 4$

For $r=2$: $(A - 2I)\vec{a} = \vec{0}$

$$\left[\begin{array}{cc|c} 5-2 & -1 & 0 \\ 3 & 1-2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Trick: $\left[\begin{array}{cc|c} A & B & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\begin{pmatrix} B \\ -A \end{pmatrix}, \begin{pmatrix} -B \\ A \end{pmatrix}$ are non-zero vector sol^{n}s.

Get $\vec{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

$$\boxed{\vec{x}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}}$$

For $r=4$: $\left[\begin{array}{cc|c} 5-4 & -1 & 0 \\ 3 & 1-4 & 0 \end{array}\right]$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$$\sim \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

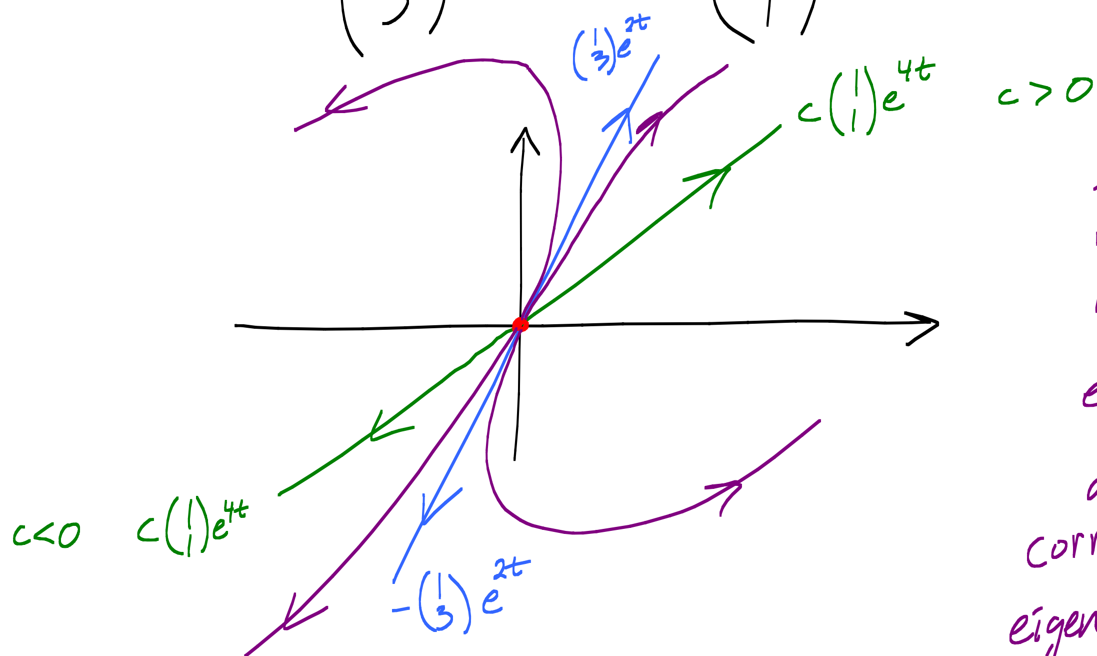
$\begin{pmatrix} B \\ -A \end{pmatrix}$ or $\begin{pmatrix} -B \\ A \end{pmatrix}$

Get $\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Get $\vec{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

(0,0) is a critical point called an "improper node"

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$



Node rule

Purple sol^{n}s hug ± the eigenvector direction corresponding to the eigenvalue closest to zero.

(Far away, the trajectories look parallel to the other eigendirections.)

Why: $\vec{x} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

$$= \left[c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}} \right] e^{2t}$$

as $t \to -\infty$

negligible compared to