Lesson 27 Linear systems with constant coefficients (homogeneous case)

HWK 10 due Wed

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11} x_1 + a_{12} x_2 \\[2ex] \dfrac{dx_2}{dt} = a_{21} x_1 + a_{22} x_2 \end{cases} \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \vec{x}\,' = \mathbb{A}\vec{x}$$

Try $\vec{x} = \vec{a} e^{rt}$ $\qquad \vec{x}\,' = r\vec{a}e^{rt} \overset{\text{want}}{=} \mathbb{A}\vec{a}e^{rt}$

$$\boxed{r\vec{a} = \mathbb{A}\vec{a}}$$

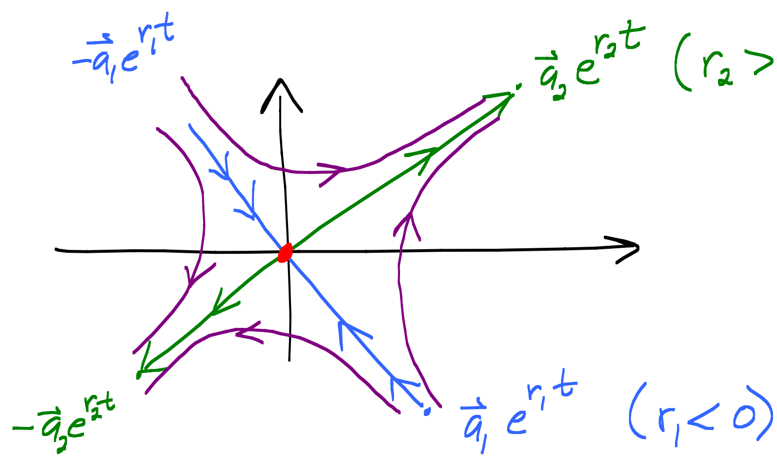
$$\underbrace{(\mathbb{A} - r\mathbb{I})}_{\text{need det } = 0}\vec{a} = \vec{0}$$

Characteristic equ: $\det(\mathbb{A} - r\mathbb{I}) = 0 \leftarrow$ 2x2 : quadratic equ

Cases: 1) r_1, r_2 real and opposite sign : $r_1 < 0 < r_2$

Get eig. vects : $\vec{a}_1, \vec{a}_2$. $\qquad (\mathbb{A} - r_k\mathbb{I})\vec{a}_k = \vec{0}$

Gen^l sol^n : $\vec{x} = c_1\vec{a}_1 e^{r_1 t} + c_2 \vec{a}_2 e^{r_2 t}$



Saddle point

Unstable

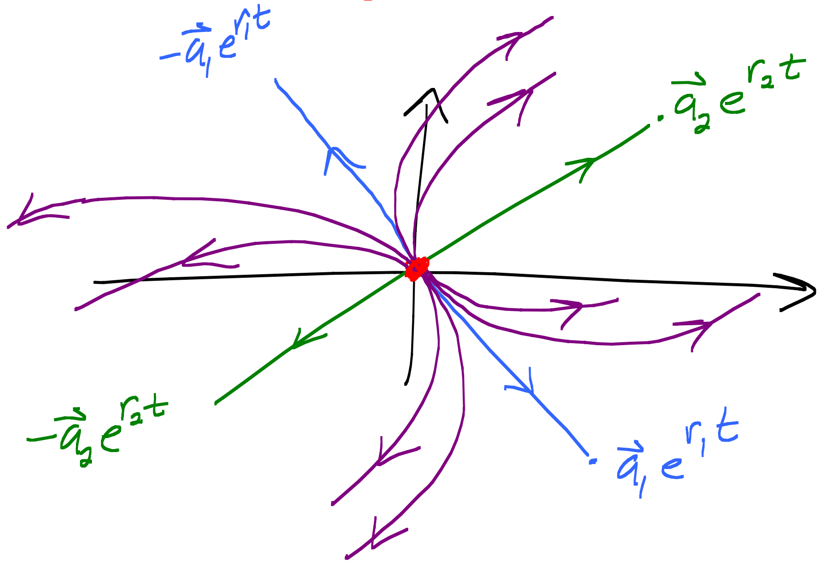
Saddle rule: Trajectories come in asymtotically to the $\pm$ e. vector directions of e. vect corresponding to the negative e. val and go out asymp to $\pm$ e vect dirs of the positive e. val.

Case 2: r_1, r_2 real and $\neq$ and same sign.

Subcase $0 < r_1 < r_2$

e.val closest to zero

$$\vec{x} = c_1 \vec{a}_1 e^{r_1 t} + c_2 \vec{a}_2 e^{r_2 t}$$



Improper node

Unstable

Node rule: Trajectories hug the $\pm$ e. vector directions near the origin corresponding to the e. value closest to zero,

Node rule part 2: Far from the origin, trajectories look parralel to the othe e. vector dir.

Subcase: $r_2 < r_1 < 0$

r_1 closest to zero

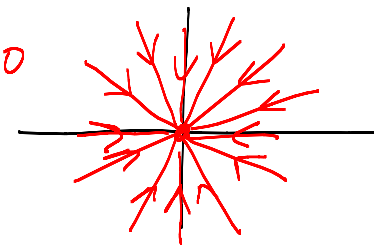
Same picture! Same Node rule. But all arrows go in. Improper node. Asymptotically stable.

EX: A proper node:

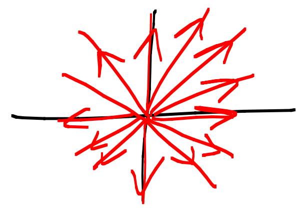
$$\begin{cases} \dfrac{dx_1}{dt} = k x_1 \\ \dfrac{dx_2}{dt} = k x_2 \end{cases}$$

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{kt} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{kt}$$

$K<0$



asymp. stable



$K>0$ Proper node

Unstable

Case $r = a \pm bi$ complex

For $r = a+bi$. Get complex e.vect $\vec{A} + i\vec{B}$.

Complex sol^n $(\vec{A}+i\vec{B})\, e^{(a+bi)t}$

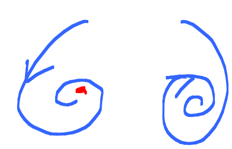
$$= \underbrace{\left[\vec{A}\cos bt - \vec{B}\sin bt\right] e^{at}}_{\vec{x}_1} + i \underbrace{\left[\vec{A}\sin bt + \vec{B}\cos bt\right] e^{at}}_{\vec{x}_2}$$

Gen^l Sol^n $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$

Critical thing: Sign of $a = \operatorname{Re} r$.

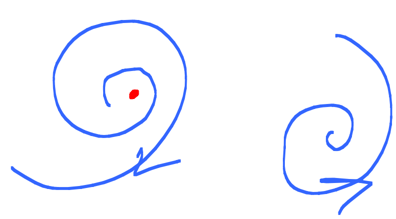
Subcase $a<0$: Type: Spiral (in)

Stability: Asympt. stable



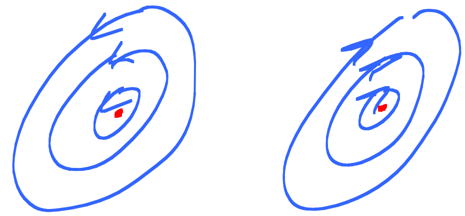
Subcase $a>0$: Spiral (out)

Unstable.



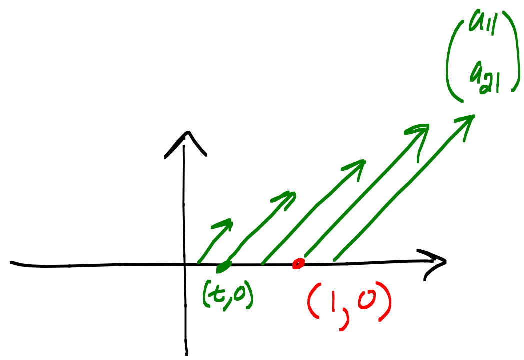
Subcase $a = 0$: Type: Center

Stability: Stable



CW or CCW? Test the field at $(1,0)$

$\vec{x}\,' = A\vec{x}$



$$\vec{x}\,'\Big|_{\text{at }(1,0)} = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

$a_{21} > 0$ CCW

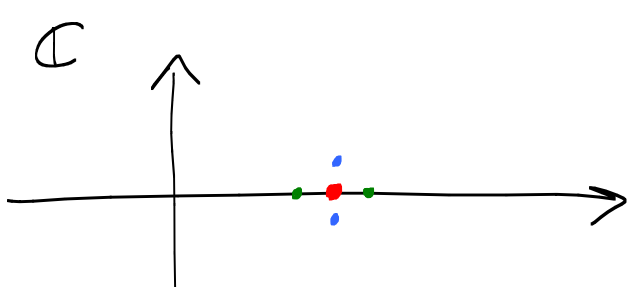
$a_{21} < 0$ CW

Spiral rule: Sign of a_{21} determines CW or CCW.

What about $r_1 = r_2$ cases? or r's $= 0$ cases.

Very delicate. Engineers try to avoid them!

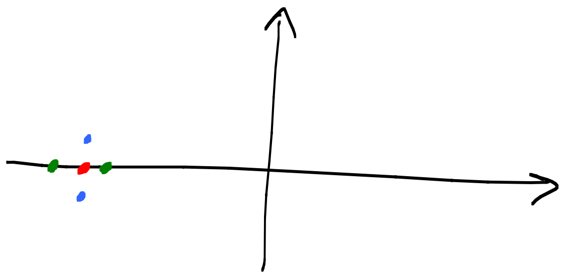
Quadratic eqn: $r = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ $\boxed{b^2 - 4ac = 0}$ $r_1 = r_2$



red dot $= \dfrac{-b}{2a}$

Improper node. Out. Unstable r_1, r_2 $\underbrace{b^2-4ac}_{\text{small}} > 0$

Spiral out. Unstable r_1, r_2 $b^2-4ac < 0$, small



Improper node. In.
Asymp. Stable

Spiral in.
Asymp. Stable

Next time: 3x3 example

$$\begin{cases} \dfrac{dx_1}{dt} = 4x_2 \\[2ex] \dfrac{dx_2}{dt} = -x_1 \\[2ex] \dfrac{dx_3}{dt} = x_1 + 4x_2 - x_3 \end{cases}$$

$$\vec{x}\,' = \begin{bmatrix} 0 & 4 & 0 \\ -1 & 0 & 0 \\ 1 & 4 & -1 \end{bmatrix} \vec{x}$$

expand along col 3

$$\det(A - r\mathbb{I}) = \det \begin{bmatrix} -r & 4 & 0 \\ -1 & -r & 0 \\ 1 & 4 & -1-r \end{bmatrix}$$