Lesson 28 3x3 system, Fundamental matrices Hwk 11 due Wed 1

$$\vec{x}' = \begin{bmatrix} 0 & 4 & 0 \\ -1 & 0 & 0 \\ 1 & 4 & -1 \end{bmatrix} \vec{x} \qquad \left(\text{Try } \vec{x} = \vec{a} e^{rt} \right)$$

$$\det(\mathbb{A} - r\mathbb{I}) = 0$$

$$\det \begin{bmatrix} 0-r & 4 & 0 \\ -1 & 0-r & 0 \\ 1 & 4 & -1-r \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= (+1)(-1-r) \det \begin{bmatrix} -r & 4 \\ -1 & -r \end{bmatrix}$$

$$= -(r+1)\left[r^2+4\right] = 0 \qquad \text{Roots } r = -1, \pm 2i$$

e.vals

For $r=-1$: $(\mathbb{A} - (-1)\mathbb{I})\vec{a} = \vec{0}$

$$\left[\begin{array}{ccc|c} 0-(-1) & 4 & 0 & 0 \\ -1 & 0-(-1) & 0 & 0 \\ 1 & 4 & -1-(-1) & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} R_2 + R_1 \\ R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \tfrac{1}{5}R_2$$

$\uparrow$ a_1 bound var $\qquad \uparrow$ a_2 bound var $\qquad \uparrow$ a_3 is a <u>free</u> variable

$\leftarrow$ Reduced Row Echelon Form (unique)

Let $a_3 = c$, anything.

$$a_1 = 0$$
$$a_2 = 0$$
$$a_3 = c$$

$$\vec{a} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} c$$

$\leftarrow$ all possible e.vectors for e.val $r=-1$ are $c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $c \neq 0$.

Take $c=1$. Get $\boxed{\vec{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t}}$

<u>For $r = 2i$</u>: $(A - (2i)I)\vec{a} = \vec{0}$

$$\left[\begin{array}{ccc|c} -2i & 4 & 0 & 0 \\ -1 & -2i & 0 & 0 \\ 1 & 4 & -1-2i & 0 \end{array}\right] \leftarrow 2i\,(R_2)$$

eliminate R_1.
Move row of zeroes to bottom

$$\sim \left[\begin{array}{ccc|c} 1 & 2i & 0 & 0 \\ 1 & 4 & -1-2i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2i & 0 & 0 \\ 0 & 4-2i & -1-2i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\cdots \sim \begin{array}{cccc} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -i/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} a_1 & a_2 & a_3 \\ \text{bound} & \text{bound} & \text{free} \end{array} \end{array} \qquad \begin{array}{l} a_1 - a_3 = 0 \\ a_2 - \frac{i}{2} a_3 = 0 \end{array}$$

Let $a_3 = c$.

$$\begin{cases} a_1 = a_3 = c \\ a_2 = \frac{i}{2} a_3 = \frac{i}{2} \cdot c \\ a_3 = c \end{cases}$$

$\vec{a} = \begin{pmatrix} 1 \\ i/2 \\ 1 \end{pmatrix} c$. Take $c = 2$.

Get <u>complex</u> sol^n $\quad \begin{pmatrix} 2 \\ i \\ 2 \end{pmatrix} e^{2it} =$

$$\left[\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} i \right] \left(\cos 2t + i \sin 2t \right)$$

$$\Bigg(\underbrace{\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \cos 2t \underset{\uparrow\, i^2}{-} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \sin 2t}_{\vec{x}_2} \Bigg) + i \Bigg(\underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \sin 2t}_{\vec{x}_3} \Bigg)$$

Get $\underline{\underline{\text{two}}}$ real sol^{n}s from one complex sol^n.

Gen^l Sol^n

$$\boxed{\vec{x} = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2\cos 2t \\ -\sin 2t \\ 2\cos 2t \end{pmatrix} + c_3 \begin{pmatrix} 2\sin 2t \\ \cos 2t \\ 2\sin 2t \end{pmatrix}}$$

Fundamental matrix

$$\mathbb{X} = \begin{bmatrix} 0 & 2\cos 2t & 2\sin 2t \\ 0 & -\sin 2t & \cos 2t \\ e^{-t} & 2\cos 2t & 2\sin 2t \end{bmatrix}$$

$$W(0) = \det \mathbb{X}(0) = \det \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= (-1) \cdot 1 \cdot \det \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} = -(-2) = 2 \leftarrow \text{not zero}$$

So W is never zero. ← Guaranteed by method.

Solving IVP's : $\vec{x}(0) = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2 + c_3 \vec{x}_3 = [\vec{x}_1, \vec{x}_2, \vec{x}_3] \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\vec{x}(0) = \underbrace{\mathbb{X}(0)}_{W(0) = \det \mathbb{X}(0) \neq 0} \vec{c} \overset{\text{want}}{=} \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

$\neq 0$. $\exists$ unique $\vec{c}$ sol^n.

Hmmm. $\vec{c} = X(0)^{-1}\begin{pmatrix}A\\B\\C\end{pmatrix}$

... and $\vec{x} = X\vec{c} = \underbrace{X\,X(0)^{-1}}_{\Phi}\begin{pmatrix}A\\B\\C\end{pmatrix}$

$\Phi \leftarrow$ a very nice fund. matrix.

Normalized Fundamental matrix: $\Phi = X\,X(0)^{-1}$

has the property: $\Phi(0) = I$

Hard way to get Φ: $\vec{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\vec{e}_2\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\vec{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

Solve IVP's $\vec{\varphi}_1(0) = \vec{e}_1$

$\vec{\varphi}_2(0) = \vec{e}_2$

$\vec{\varphi}_3(0) = \vec{e}_3$

Then $\Phi = [\vec{\varphi}_1, \vec{\varphi}_2, \vec{\varphi}_3]$

$\vec{x} = X\vec{c} = \vec{e}_k$ $k=1,2,3$

$\left[X(0)\,\middle|\,\vec{e}_1, \vec{e}_2, \vec{e}_3\right]$

$\rightsquigarrow \left[I\,\middle|\,X(0)^{-1}\right]$

Aha! $\Phi = X \cdot X(0)^{-1}$

2x2 case is easy: $\begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

Virtue of normalized F.M.: $\vec{x} = \Phi\vec{c}$ is gen^l sol^n.

IVP: $\vec{x}(0) = \Phi(0)\vec{c} = \begin{pmatrix}A\\B\\C\end{pmatrix}$

$\text{II}\ \vec{c} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$

$\vec{c} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$ easy!

Abel's formula:

$$\begin{cases} \dfrac{dx_1}{dt} = p_{11}(t)\, x_1 + p_{12}(t)\, x_2 \\[2ex] \dfrac{dx_2}{dt} = p_{21}(t)\, x_1 + p_{22}(t)\, x_2 \end{cases}$$

Two sol^n's $\quad \vec{x}_1(t) = \begin{pmatrix} x_{11}(t) \\ x_{12}(t) \end{pmatrix} \qquad \vec{x}_2(t) = \begin{pmatrix} x_{21}(t) \\ x_{22}(t) \end{pmatrix}$

$$W = \det \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{bmatrix}$$

kill via p_{12} (Row 2)

$$\frac{dW}{dt} = \det \begin{bmatrix} x_{11}' & x_{21}' \\ x_{12} & x_{22} \end{bmatrix} + \det \begin{bmatrix} x_{11} & x_{21} \\ x_{12}' & x_{22}' \end{bmatrix}$$

$x_{11}' = p_{11}x_{11} + p_{12}x_{21}$, $\quad x_{21}' = p_{11}x_{21} + p_{12}x_{22}$

$x_{12}' = \underline{\underline{p_{21}x_{11}}} + p_{22}x_{12}$, $\quad x_{22}' = \underline{\underline{p_{21}x_{21}}} + p_{22}x_{22}$

kill via p_{21} (Row 1)

$$= p_{11} W + p_{22} W$$

Abel's formula $\quad \boxed{\dfrac{dW}{dt} = (p_{11} + p_{22})\, W}$

Sol^n: $W = Ce^{\int p_{11} + p_{22}\, dt}$

N-th order linear eqn $\rightsquigarrow$ $N \times N$ 1st order system

Abel's $=$ Abel's