Lesson 29 Repeated roots

Hwk 11 due Wed. Exam 2 in class Wed., Nov. 13

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\frac{d}{dx}(e^x) = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots = e^x$$

$$\frac{d}{dx}\left(e^{cx}\right) = c\left(e^{cx}\right) \leftarrow \text{solves } f' = cf$$

Systems: $\vec{x}' = \mathbb{A}\vec{x}$

Hmmm: $\mathbb{I} + \mathbb{A}t + \frac{1}{2!}\mathbb{A}^2t^2 + \frac{1}{3!}\mathbb{A}^3t^3 + \cdots = \mathbb{X}$

$\uparrow$ $\exp(\mathbb{A}t)$

Show $\mathbb{X}' = \mathbb{A}\mathbb{X}$

$\vec{x} = \mathbb{X}\vec{c}$ is a gen^l sol^n

Repeated roots:

$$\vec{x}\,' = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}\vec{x}$$

$$\det\left(\mathbb{A} - r\mathbb{I}\right) = 0$$

$$\det\begin{bmatrix} -3-r & 1 \\ -1 & -1-r \end{bmatrix} = (-3-r)(-1-r) + 1$$

$$= r^2 + 4r + 4$$

$$= (r+2)^2 = 0$$

Ouch! Only one e.val. $r = -2, -2 \leftarrow$ $r=-2$ is an e.val of algebraic multiplicity 2

For $r=-2$: $\left(\mathbb{A} - (-2)\mathbb{I}\right)\vec{a} = \vec{0}$

$$\left[\begin{array}{cc|c} -3-(-2) & 1 & 0 \\ -1 & -1-(-2) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \leftarrow a_1 = \underset{\substack{\uparrow \\ \text{free}}}{a_2}$$

Get $\vec{a} = \begin{pmatrix} c \\ c \end{pmatrix}$. Take $c=1$. $\boxed{\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

Only get one sol^n to system : $\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$

Can use high school method. Fun. Leads you to guess:

Maybe $\vec{x}_2 = t \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$. $\longleftarrow$ Nope!

Next: Try $\boxed{\vec{x}_2 = \vec{a}\, t e^{-2t} + \vec{b} e^{-2t}}$

$$\vec{x}\,' = \vec{a}\left(1 \cdot e^{-2t} - 2te^{-2t}\right) + (-2)\vec{b} e^{-2t}$$

$$\underbrace{-2\vec{a}\,te^{-2t} + \left(\vec{a} - 2\vec{b}\right)e^{-2t}}_{\vec{x}\,'} \underset{\substack{\uparrow \\ \text{want}}}{=} \underbrace{\mathbb{A}\vec{a}\,te^{-2t} + \mathbb{A}\vec{b} e^{-2t}}_{\mathbb{A}\vec{x}}$$

$$\underbrace{\left[-2\vec{a} - A\vec{a}\right]}_{\text{need } = 0} te^{-2t} + \underbrace{\left[\vec{a} - 2\vec{b} - A\vec{b}\right]}_{=0} e^{-2t} = \vec{0}$$

$$A\vec{a} = -2\vec{a}$$

Aha! $\vec{a}$ = e.vect for $r = -2$

Take $\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\underbrace{(A - (-2)I)}_{\text{Danger!}}\vec{b} = \vec{a}$$

Danger!
det = 0
Maybe impossible to find $\vec{b}$.
Miracle: Can find $\vec{b}$ when r is repeated.

Recap: r repeated root.

Get $\vec{x}_1 = \vec{a}e^{rt}$ where $\vec{a}$ is e.vect for e.val r.

Get $\vec{x}_2 = \vec{a}te^{rt} + \vec{b}e^{rt}$ where $\vec{a}$ is as above and

$$\boxed{(A - rI)\vec{b} = \vec{a}}$$

← guaranteed to have a sol^n

EX:

$$\left[\begin{array}{cc|c} -3-(-2) & 1 & 1 \\ -1 & -1-(-2) & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} \leftarrow b_1 = b_2 - 1 \\ \leftarrow \text{Danger spot. Safe!} \end{array}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ b_1 \text{ bound} & b_2 \text{ free} \end{array}$$

Let $b_2 = c$. $\quad b_1 = b_2 - 1 = c - 1$

$$b_2 = c$$

So $\vec{b} = \begin{pmatrix} c-1 \\ c \end{pmatrix} = \underbrace{\begin{pmatrix} c \\ c \end{pmatrix}}_{\text{e. vector!}} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\leftarrow$ Just need one $\vec{b}$ that works. Take $c = 0$.

$$\vec{b} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Get $\boxed{\vec{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-2t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{-2t}}$

$$\mathbb{X} = [\vec{x}_1, \vec{x}_2] = \begin{bmatrix} e^{-2t} & (t-1)e^{-2t} \\ e^{-2t} & t e^{-2t} \end{bmatrix}$$

$W = \det \mathbb{X} = e^{-4t}$ $\leftarrow$ not zero (Abel's never zero)

So $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} t-1 \\ t \end{pmatrix} e^{-2t}$ is gen'l sol'n

EX:
$$\begin{cases} \dfrac{dx_1}{dt} = x_1 \\[2ex] \dfrac{dx_2}{dt} = x_2 \end{cases}$$

$$\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$$

$$\det\left(\mathbb{I} - r\,\mathbb{I}\right) = 0$$

$$x_1 = c_1 e^t$$

$$x_2 = c_2 e^t$$

$$\vec{x} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^t = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$$

$$(1-r)^2 = 0$$

$$r = 1, 1$$

$r=1$ e.val of algebraic mult. 2.

$$(\mathbb{I} - 1\,\mathbb{I})\vec{a} = 0$$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Whoa! Eigenspace of all e.vect (plus $\vec{0}$) for $r=1$ is $\mathbb{R}^2 \leftarrow$ 2 dim^{l}

__Def^n:__ Multiplicity of the root of $\det(\mathbb{A} - r\mathbb{I}) = 0$ is called the __algebraic multiplicity__ of the e.val. r. The dimension of the eigenspace for r is called the __geometric multiplicity__.

__Fact:__ (geometric mult) $\leq$ (algebraic mult).

Back to example:

$$\vec{x} = c_1 \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{a}_1} e^t + c_2 \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{a}_2} e^t$$

Two linearly indep e.vectors yield __two__ linearly indep sol^{n}s to system.

__Non-homogeneous systems__ 7.9 $\quad \vec{x}' = \mathbb{A}\vec{x} + \underbrace{\vec{g}}_{\text{not zero}}$

Theorem:

$\mathbb{A}$ is $n \times n$

$$\vec{x}\,' = \mathbb{A}\vec{x} \leftarrow (*) \quad \text{homogeneous linear syst}$$

$$\vec{x}\,' = \mathbb{A}\vec{x} + \vec{g} \quad (+) \quad \text{non-homog}$$

Gen^l sol^n to $(+)$ is

$$\vec{x} = \underbrace{\left(c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n\right)}_{\text{Gen}^l\ \text{Sol}^n\ \text{to}\ (*)} + \underset{\substack{\uparrow \\ \text{one particular} \\ \text{sol}^n\ \text{to}\ (+)}}{\vec{x}_p}$$

EX:

$$\vec{x}\,' = \underbrace{\begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}}_{\substack{\text{Got gen}^l \\ \text{sol}^n\ \text{to}\ (*)\ \text{above}}} \vec{x} + \begin{pmatrix} 2e^{3t} \\ e^{3t} \end{pmatrix}$$

Try $\vec{x}_p = \vec{b}\,e^{3t}$

Make it work.

$r \neq -2 \leftarrow$ e.val

makes it safe!