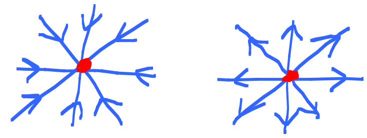
Lesson 30 Nonhomogeneous first order linear systems

Repeated roots: $r_1 = r_2$, 2 linearly independent e.vectors

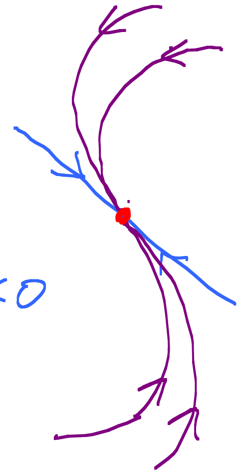
Proper nodes



r's < 0 $\qquad$ r's > 0

$r_1 = r_2$, only one e.vector

Improper node



r's < 0

r's > 0: Arrows go out instead of in

Nonhomogeneous systems:

$$\vec{x}' = \mathbb{P}\vec{x} + \vec{g}(t) \quad (+), \qquad \mathbb{P} = \begin{bmatrix} \text{matrix of} \\ \text{fcns of } t \end{bmatrix}_{n\times n}$$

Gen' Sol^n:
$$\vec{x} = \underbrace{\left(c_1\vec{x}_1 + \cdots + c_n\vec{x}_n\right)}_{\substack{\text{Gen' sol}^n \text{ to} \\ (*)\ \vec{x}' = \mathbb{P}\vec{x}\ \text{(homog)}}} + \underbrace{\vec{x}_p}_{\text{just one part. sol}^n!}$$

Why: Suppose $\vec{x}$ solves $(+)$.

$$\begin{aligned} \vec{x}' &= \mathbb{P}\vec{x} + \vec{g} \\ - \quad \vec{x}_p' &= \mathbb{P}\vec{x}_p + \vec{g} \\ \hline \end{aligned}$$

Aha!
$$\underbrace{\vec{x}' - \vec{x}_p'}_{(\vec{x}-\vec{x}_p)'} = \mathbb{P}\vec{x} - \mathbb{P}\vec{x}_p = \mathbb{P}(\vec{x} - \vec{x}_p)$$

See $\vec{x} - \vec{x}_p$ solves homog!

So $\vec{x} - \vec{x}_p = \underbrace{c_1\vec{x}_1 + \cdots + c_n\vec{x}_n}_{\vec{x}_c}$ for some choice of c's.

So $\vec{x} = \vec{x}_c + \vec{x}_p$.

Last thing to check: Anything of form $\left(c_1\vec{x}_1 + \cdots + c_n\vec{x}_n\right) + \vec{x}_p$ solves $(+)$. Easy. Just plug it in.

EX:

$$\begin{cases} \dfrac{dx_1}{dt} = -3x_1 + x_2 + e^{2t} \\[2ex] \dfrac{dx_2}{dt} = x_1 - 3x_2 + 2e^{2t} \end{cases}$$

$$\vec{x}\,' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \vec{x} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$$

$$\vec{x}\,' = \mathbb{A}\vec{x} + \vec{g}$$

Step 1: Solve homog. $\vec{x}\,' = \mathbb{A}\vec{x}$ $\quad \det(\mathbb{A} - r\mathbb{I}) = 0 \quad r_1, r_2$

$$(\mathbb{A} - r_k\mathbb{I})\vec{a}_k = \vec{0} \quad k = 1, 2$$

Get $\boxed{\vec{x}_c = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}}$ $\quad r = -2, -4$

Step 2: Find one $\vec{x}_p$. $\quad \vec{g} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} \leftarrow$ whew! 2 is not an eval.

Can use method of Undet Coeff.

Guess $\boxed{\vec{x}_p = \vec{b}\, e^{2t}}$ $\quad x_1 = b_1 e^{2t}$, $\quad x_2 = b_2 e^{2t}$

Long way: Plug into long-hand form of system. Divide out e^{2t}'s. Get linear alg prob.

Quick way: $\vec{x} = \vec{b}\, e^{2t}$

Want

$$(\vec{b}\, e^{2t})' = \mathbb{A}(\vec{b}\, e^{2t}) + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$$

$$2\vec{b}\, e^{2t} = (\mathbb{A}\vec{b})e^{2t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$$

Divide out e^{2t}s:

$$\boxed{2\vec{b} = \mathbb{A}\vec{b} + \binom{1}{2}}$$

$$\begin{cases} 2b_1 = -3b_1 + b_2 + 1 \\ 2b_2 = b_1 - 3b_2 + 2 \end{cases}$$

$$-\binom{1}{2} = \mathbb{A}\vec{b} - 2\vec{b}$$

$$= (\mathbb{A} - 2\mathbb{I})\vec{b}$$

✓

$$\begin{cases} -1 = (-3-2)b_1 + b_2 \\ -2 = b_1 + (-3-2)b_2 \end{cases}$$

$$(\mathbb{A} - 2\mathbb{I})\vec{b} = -\binom{1}{2}$$

not e.val

So det $\neq 0$.

Unique sol^n $\vec{b}$.

$$\left[\begin{array}{cc|c} -3-2 & 1 & -1 \\ 1 & -3-2 & -2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -5 & 1 & -1 \\ 1 & -5 & -2 \end{array}\right]$$

Use row reduction.
or Cramer's rule

Get $b_1 = \frac{7}{24}$, $b_2 = \frac{11}{24}$

Gen^l sol^n to $(+)$: $\vec{x} = c_1\binom{1}{1}e^{-2t} + c_2\binom{1}{-1}e^{-4t} + \frac{1}{24}\binom{7}{11}e^{2t}$

Method of Undetermined Coeff:

For solving $\vec{x}' = \mathbb{A}\vec{x} + \vec{g}$

$\vec{g}$ special

const. coeff

$\vec{g}$	Try $\vec{x}_p$
$\vec{A}e^{rt}$	$\vec{b}e^{rt}$ $\leftarrow$ danger if r eval for A
$\vec{A}t^n$ or anything vector of deg n.	$\vec{b}_n t^n + \vec{b}_{n-1} t^{n-1} + \vec{b}_{n-2} t^{n-2} + \cdots + \vec{b}_1 t + \vec{b}_0$ (danger if $r=0$ is an e.val)
$\vec{A}\cos\omega t$ or $\vec{B}\sin\omega t$ or a combo	$\vec{x} = \vec{a}\cos\omega t + \vec{b}\sin\omega t$ $\leftarrow$ danger $r = \pm\omega i$ e.val's.
$\vdots$ $\vec{A}t^n e^{rt}\cos\omega t$	Guess, do it.

When in danger, use variation of parameters:

$$\vec{x}\,' = \mathbb{P}(t)\,\vec{x} + \vec{g}$$

$\mathbb{P}(t)$: matrix of fcns of t

$\vec{g}$: not special

Step 1: (Hard part). Solve homog $\vec{x}\,' = \mathbb{P}\vec{x}$ (*).

Get $\vec{x}_c = c_1\vec{x}_1 + \cdots + c_n\vec{x}_n$ $\leftarrow$ c's consts.

Step 2: Find a part sol^n $\vec{x}_p$ of form

$\vec{x}_p = u_1\vec{x}_1 + \cdots + u_n\vec{x}_n$ $\leftarrow$ u's fcns of t.

Plug in. Force it.

$$\vec{x}_p' = u_1'\vec{x}_1 + \underline{u_1\vec{x}_1'} + \cdots + u_n'\vec{x}_n + \underline{u_n\vec{x}_n'} \overset{\text{want}}{=} \mathbb{P}(u_1\vec{x}_1 + \cdots + u_n\vec{x}_n) + \vec{g}$$

$$= \underline{u_1(\mathbb{P}\vec{x}_1)} + \cdots + \underline{u_n(\mathbb{P}\vec{x}_n)} + \vec{g}$$

$$u_1'\vec{x}_1 + \cdots + u_n'\vec{x}_n = \vec{g}$$

$$\underbrace{[\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n]}\begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix} = \vec{g}$$

Fund matrix: $\mathbb{X}$

Formula for u's: $\boxed{\mathbb{X}\vec{u}' = \vec{g}}$

EX: $\vec{x}' = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}\vec{x} + \begin{pmatrix} 1 \\ e^t \end{pmatrix}$

Step 1: $\vec{x}_c = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t + c_2\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{2t}$

Danger! Part of $\vec{g}$ is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^t$

Step 3: V of P to get $\vec{x}_p$

$$\mathbb{X} = \begin{bmatrix} e^t & 3e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}$$

$$\mathbb{X}\vec{u}' = \vec{g}$$

$$\begin{bmatrix} e^t & 3e^{2t} \\ e^t & 2e^{2t} \end{bmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 1 \\ e^t \end{pmatrix}$$

Cramer's rule:

$$u_1' = \frac{\det \begin{bmatrix} 1 & 3e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}}{\det \begin{bmatrix} e^t & 3e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}} = -2e^{-t} + 3$$

$$u_2' = -e^{-t} + e^{-2t}$$

Finally,

$$\begin{cases} u_1 = \int -2e^{-t} + 3 \, dt = 2e^{-t} + 3t \\ u_2 = e^{-t} - \frac{1}{2}e^{-2t} \end{cases}$$

no $+C$'s

Get $\vec{x}_p = u_1 \vec{x}_1 + u_2 \vec{x}_2$

$$= (2e^{-t} + 3t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \left(e^{-t} - \frac{1}{2}e^{-2t}\right)\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{2t}$$