Lesson 31 Non-linear 2x2 systems 9.1, 9.2

Chap 3, 4 → Exam 2 in class Wed., Nov. 13

Practice problems on home page

HWK 12 due Friday, Nov. 15

Good practice probs on Chap 7

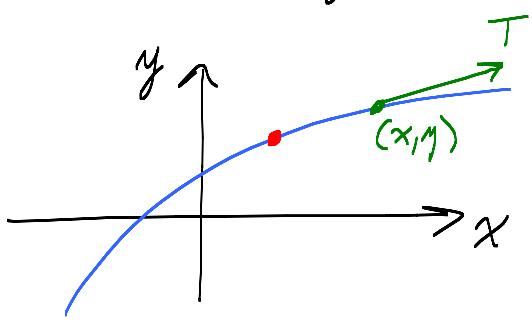
Autonomous non-linear 2x2 first order systems. EX: $\vec{x}' = \mathbb{A}\vec{x}$

$$\begin{cases} f(x,y) = a_{11}x + a_{12}y \\ g(x,y) = a_{21}x + a_{22}y \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases}$$

autonomous: no t in f or g.

Phase plane: Sol^n $(x(t), y(t))$ curve in $\mathbb{R}^2$.



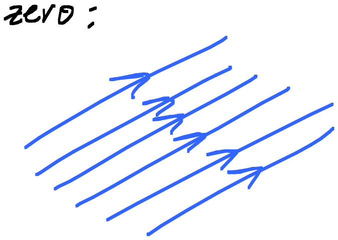
$x(t_0) = x_0$

$y(t_0) = y_0$

$$T = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \underbrace{\big(f(x,y), g(x,y)\big)}_{\text{direction field}}$$

Look at sol^n's under a magnifying glass:

Case $f(x_0,y_0)$, $g(x_0,y_0)$ not both zero:



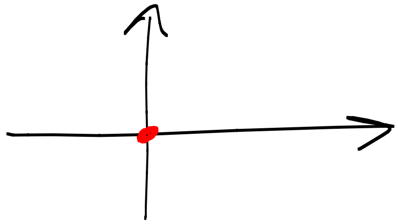
Case $f(x_0,y_0) = 0$, $g(x_0,y_0) = 0$ Critical point.

Very interesting!

How to linearize a system at a critical point

Supose crt pt is $(0,0)$. $f(0,0) = 0$, $g(0,0) = 0$.

<u>Note</u>: $x(t) \equiv 0$, $y(t) \equiv 0$ solves syst. ✓



<u>Idea</u>: Use Taylor's formula

$$\begin{cases} f(x,y) = \underbrace{f(x_0,y_0)}_{=0} + \frac{\partial f}{\partial x}(x_0,y_0)(x - \overset{0}{\overset{\shortparallel}{x_0}}) + \frac{\partial f}{\partial y}(x_0,y_0)(y - \overset{0}{\overset{\shortparallel}{y_0}}) + R_f(x,y) \\ g(x,y) = \underbrace{g(x_0,y_0)}_{=0} + g_x^0 \cdot (x - x_0) + g_y^0 \cdot (y - y_0) + R_g(x,y) \end{cases}$$

$$\begin{cases} \frac{dx}{dt} = f_x^0 \cdot x + f_y^0 \cdot y + R_f(x,y) \\ \frac{dy}{dt} = g_x^0 \cdot x + g_y^0 \cdot y + R_g(x,y) \end{cases}$$

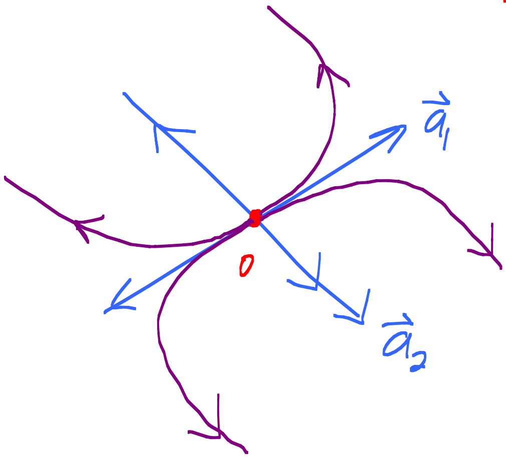
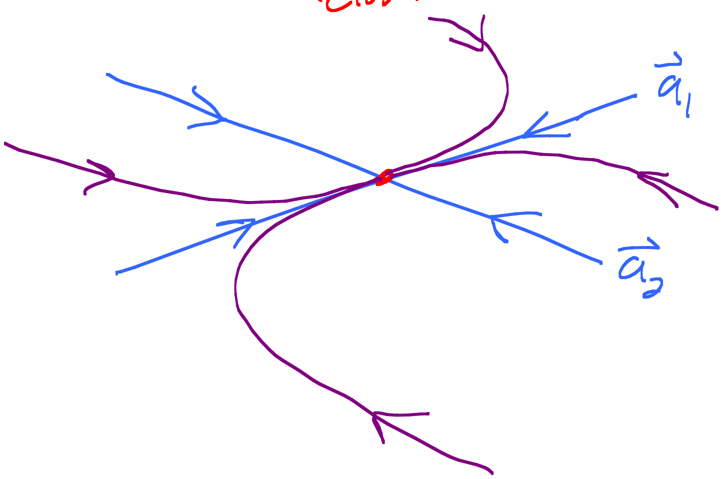
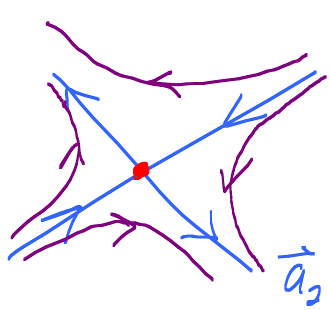
R's negligible under magnification

Write $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$.

Get $\vec{x}\,' = \mathbb{A}\vec{x}$ where $\mathbb{A} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}_{(x_0,y_0)}$

<u>Great fact</u>: Analyze $\vec{x}\,' = \mathbb{A}\vec{x}$. Magnified behavior near crt pt of non-linear syst is same linearized system when:

<u>Find e-vals, e-vectors</u> r_1, r_2

Linearized system	non-linear syst
<u>Case</u>: r_1, r_2 real, $0 < r_1 < r_2$ closest to zero  Improper node. unstable	Same! Same!
<u>Case</u> $r_2 < r_1 < 0$ closest to zero  Improper node. Asymt. stable	Same! Same!
<u>Case</u> $r_1 < 0 < r_2$ 	Same!

Saddle pt. Unstable	Same!
Complex roots $r = a \pm bi$ Case: $a < 0$ Spiral in. Asympt. stable cw ccw	Same!
Case: $a > 0$ Spiral out Unstable cw ccw	Same!
Case: $a = 0$ Center Stable	Center, or spiral in, or spiral out
Repeated roots: $r_1 = r_2$	Too delicate! But ...
r's < 0 Asymp stable	Same.
r's > 0 Unstable	Same.

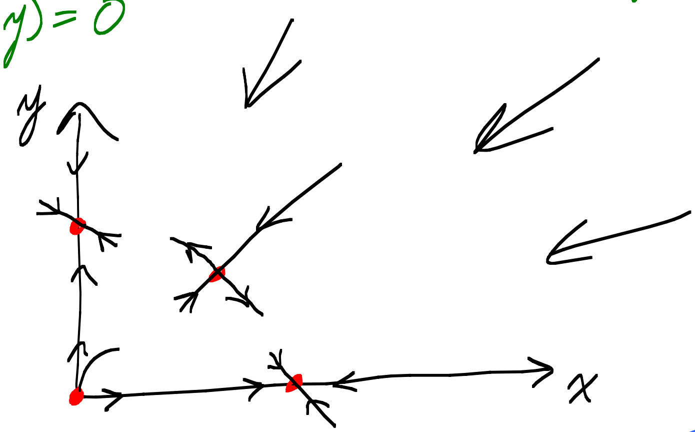
Famous problem: Competing species

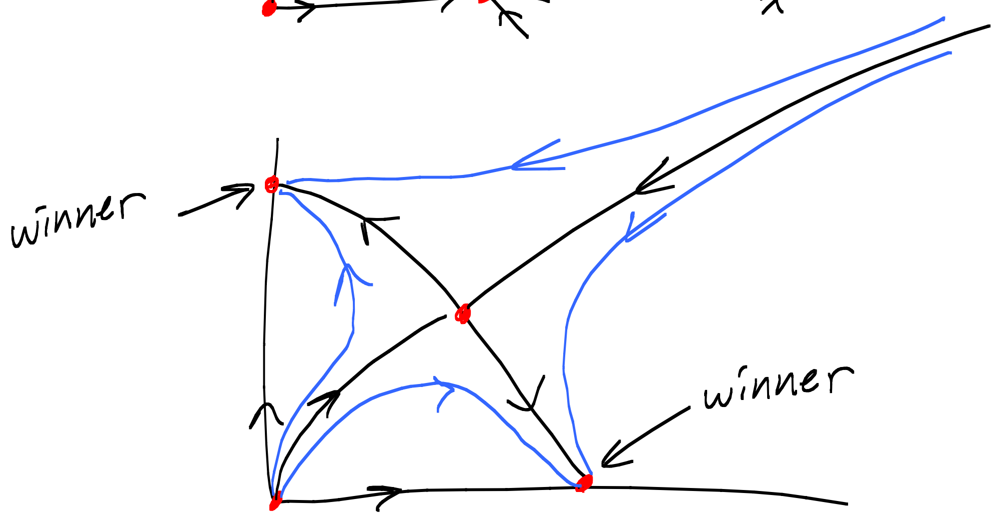
$x =$ perch

$y =$ bluegills

$$\begin{cases} \dfrac{dx}{dt} = \text{quadratic in } x, y \\[2ex] \dfrac{dy}{dt} = \text{same} \end{cases}$$

Set $\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$ Find 4 crt pts.





Important facts: $c_1\vec{x}_1 + \cdots + c_n\vec{x}_n$ gen' sol^n to syst.

$\iff W(t_0) \neq 0 \iff W$ never zero

$\iff \vec{x}_1, \ldots, \vec{x}_n$ linearly independent