Lesson 32 Competing species

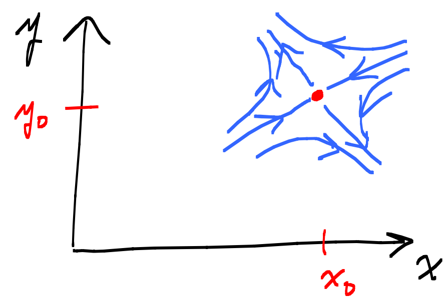
Exam 2 on Wed in class
HWK 12 due Fri after exam
Practice problems on home page, Ch. 3, 4
HWK 12 probs: good practice for Ch. 7

(x_0, y_0) Critical pt.

$$\begin{cases} \dfrac{du}{dt} = \dfrac{dx}{dt} = f(x,y) = \underbrace{f(x_0,y_0)}_{0} + \dfrac{\partial f}{\partial x}(x_0,y_0)\overbrace{(x-x_0)}^{u} + \dfrac{\partial f}{\partial y}(x_0,y_0)\overbrace{(y-y_0)}^{v} + \cancel{R_f} \\ \dfrac{dv}{dt} = \dfrac{dy}{dt} = g(x,y) = \underbrace{g(x_0,y_0)}_{0} + \dfrac{\partial g}{\partial x}(x_0,y_0)\overbrace{(x-x_0)}^{u} + \dfrac{\partial g}{\partial y}(x_0,y_0)\overbrace{(y-y_0)}^{v} + \cancel{R_g} \end{cases}$$

Linearized system: $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ $\quad \vec{u}' = A\vec{u}$ where

$$A = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}_{(x_0,y_0)}$$



(x_0, y_0) is the new origin for localized system.

Competing species:

$$\begin{cases} \dfrac{dx}{dt} = x - x^2 - xy = \underbrace{(1-x-y)}_{k_1}\cdot x \\ \dfrac{dy}{dt} = \frac{1}{2}y - \frac{1}{4}y^2 - \frac{3}{4}xy = \underbrace{\left(\frac{1}{2}-\frac{1}{4}y-\frac{3}{4}x\right)}_{k_2}\cdot y \end{cases}$$

Crt pts: $x(1-x-y) = 0 \leftarrow \quad x = 0$ or $1-x-y=0$

$y\left(\frac{1}{2}-\frac{1}{4}y-\frac{3}{4}x\right) = 0$

Case $x=0$:

$y\left(\frac{1}{2}-\frac{1}{4}y-0\right) = 0 \quad y = 0, 2$

$\boxed{\text{Get } (0,0), (0,2)}$

Case: $1-x-y=0$ (A)

(B) $y\left(\frac{1}{2}-\frac{1}{4}y-\frac{3}{4}x\right) = 0 \leftarrow y = 0$

in which case

(A) yields $1-x-0=0$

$x = 1.$

$\boxed{\text{Get } (1,0), \left(\tfrac{1}{2}, \tfrac{1}{2}\right)}$

$\underline{\underline{\text{or}}}$ $\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x = 0$ (C)

Need (A): $x + y = 1$ Get $x = \frac{1}{2}$, $y = \frac{1}{2}$

(C): $3x + y = 2$

$\underline{\text{Linearize at Crt pts}}$: $\vec{u}' = \mathbb{A}\vec{u}$ where

$$\mathbb{A} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}_{(x_0, y_0)} = \begin{bmatrix} 1 - 2x - y & -x \\ -\frac{3}{4}y & \frac{1}{2} - \frac{1}{2}y - \frac{3}{4}x \end{bmatrix}_{(x_0, y_0)}$$

$\underline{\text{At Crt pt } (0,0)}$: $\mathbb{A} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

$\det(\mathbb{A} - r\mathbb{I}) = (1-r)(\frac{1}{2} - r)$ $r = \frac{1}{2}, 1$ $0 < \frac{1}{2} < 1$

closest to 0

$\underline{\text{Unstable node}}$ $r = \frac{1}{2}$: $(\mathbb{A} - \frac{1}{2}\mathbb{I})\vec{a} = 0$

$$\left[\begin{array}{cc|c} 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \leftarrow a_1 = 0$$

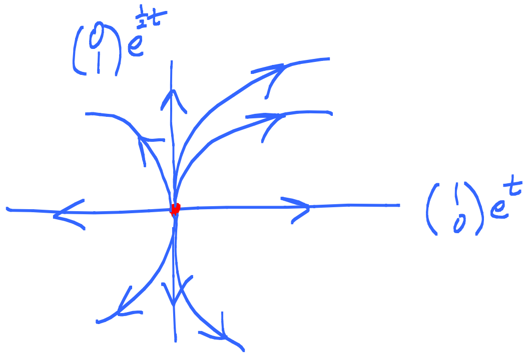
a_2 free

$\vec{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$a_2 = t$ Take $t = 1$

$\underline{r = 1}$: $\vec{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\vec{u} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\frac{1}{2}t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t}$$

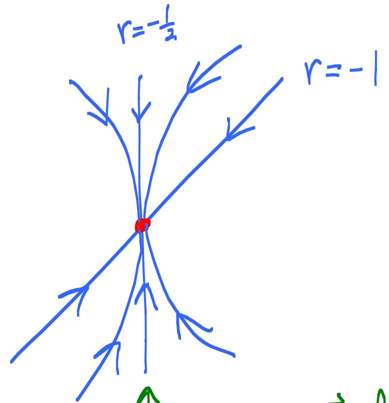


At $(0,2)$: $\mathbb{A} = \begin{bmatrix} -1 & 0 \\ -3/2 & -\frac{1}{2} \end{bmatrix}$ $\quad r = -\frac{1}{2}, -1$

closest to 0

Asymt stabe node: $\vec{u} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\frac{1}{2}t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$

red dot is $(0,2)$



hug the $\vec{a}$ for e.val closest to zero near origin

At $(1,0)$: $r = -\frac{1}{4}, -1$ $\quad$ Asymp. stabe node.

closest

At $(\frac{1}{2}, \frac{1}{2})$: $\mathbb{A} = \begin{bmatrix} -1/2 & -1/2 \\ -3/8 & -1/8 \end{bmatrix}$

$$\det \begin{bmatrix} -1/2 - r & -\frac{1}{2} \\ -3/8 & -\frac{1}{8} - r \end{bmatrix} = \left(-\tfrac{1}{2} - r\right)\left(-\tfrac{1}{8} - r\right) - \frac{3}{16} = 0$$

$$r^2 + \frac{5}{8} r - \frac{1}{8} = 0$$

$$8r^2 + 5r - 1 = 0$$

$$r = \frac{-5 \pm \sqrt{25 + 32}}{16} = -\frac{5}{16} \pm \frac{\sqrt{57}}{16}$$

One + other −

Saddle pt. Unstable!

$$r = -\frac{5}{16} - \frac{\sqrt{57}}{16}$$

$$\left[\begin{array}{cc|c} -\frac{1}{2} - \left(-\frac{5}{16} - \frac{\sqrt{57}}{16}\right) & -\frac{1}{2} & 0 \\ \text{who cares} & & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} -\frac{3}{16} + \frac{\sqrt{57}}{16} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} A & B & 0 \\ 0 & 0 & 0 \end{array}\right] \leftarrow \text{Take } \vec{a} = \begin{pmatrix} B \\ -A \end{pmatrix} \text{ or } \begin{pmatrix} -B \\ A \end{pmatrix}$$

Get $\begin{pmatrix} 1/2 \\ -3/16 + \frac{\sqrt{57}}{16} \end{pmatrix}$. Better one: $\begin{pmatrix} 8 \\ -3+\sqrt{57} \end{pmatrix}$

ETC: