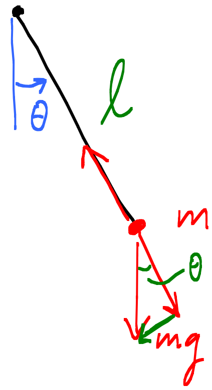
Lesson 34 Damped pendulum with a stiff string HWK 13 due Friday



$F = ma$ in tangential direction

$$mg\sin\theta = -m\ell\ddot{\theta} - c(\ell\dot{\theta})$$

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta - k\dot{\theta}$$

System: $\ddot{\theta} = -\sin\theta - \theta$ $\quad \left(\frac{g}{\ell} = 1, \ k = 1\right)$

$$\begin{cases} x(t) = \theta(t) \\ y(t) = \dot{\theta}(t) \end{cases} \qquad \begin{cases} \dfrac{dx}{dt} = \dot{\theta} = y \\ \dfrac{dy}{dt} = \ddot{\theta} = -\sin\theta - \dot{\theta} = -\sin x - y \end{cases}$$

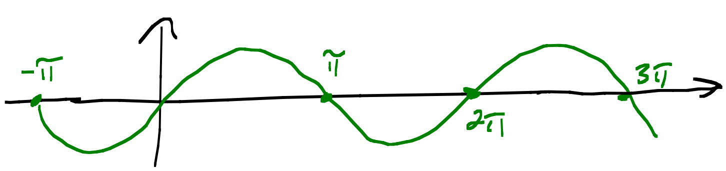
$$\begin{cases} \dfrac{dx}{dt} = y \leftarrow f(x,y) \\ \dfrac{dy}{dt} = -\sin x - y \leftarrow g(x,y) \end{cases}$$

Step 1: Find the crt pts (x_0, y_0) where

$$\begin{cases} f(x_0, y_0) = 0 & (A) \\ g(x_0, y_0) = 0 & (B) \end{cases}$$

(A): $y = 0$

(B) $-\sin x - \underbrace{y}_{=0} = 0 \Leftarrow \sin x = 0$



$$\begin{cases} y = 0 \\ x = \pm n\pi \end{cases}$$

Step 2: Linearize at Crt pts.

At (x_0, y_0), linearized system is $\vec{x}\,' = \mathbb{A}\vec{x}$

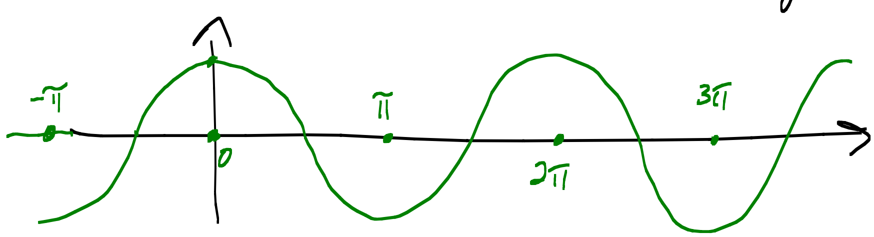
origin corresponds to Crt pt. (x_0, y_0)

where $\mathbb{A} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}\Bigg|_{(x_0,y_0)}$

$f(x,y) = y$

$g(x,y) = \sin x - y$

$= \begin{bmatrix} 0 & 1 \\ -\cos x & -1 \end{bmatrix}\Bigg|_{(x_0,y_0)}$



At <u>even</u> multiples of π, $\cos n\pi$ is $+1$.
At <u>odd</u>, $\cos n\pi$ is -1.

<u>Case:</u> n even (including $n=0$).

$\mathbb{A} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ $\quad \det(\mathbb{A} - r\mathbb{I}) = \det\begin{bmatrix} -r & 1 \\ -1 & -1-r \end{bmatrix}$

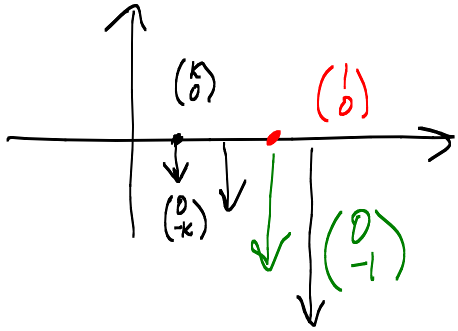
$= -r(-1-r) + 1$

$= r^2 + r + 1 = 0$

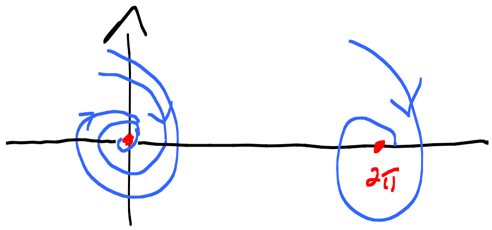
Spiral <u><u>in</u></u>
Asympt. stable $\longrightarrow$ $r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

CW or CCW? Check $\vec{x}\,'$ at $\vec{x} = \binom{1}{0}$:

$\vec{x}\,' = \mathbb{A}\binom{1}{0} = \left(\substack{\text{first} \\ \text{col of } \mathbb{A}}\right) = \binom{0}{-1}$



Clockwise!



Case Odd multiples of π. $\quad \cos n\pi = -1$

$$\mathbb{A} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \qquad \det(\mathbb{A} - r\mathbb{I}) = 0$$

$$-r(-1-r)-1 = 0$$

$$r^2 + r - 1 = 0$$

$$r = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

For $r = -\frac{1}{2} - \frac{\sqrt{5}}{2}$: $\qquad (\mathbb{A} - r\mathbb{I})\vec{a} = 0$

$r = -\frac{1}{2} - \frac{\sqrt{5}}{2}$

$$\left[\begin{array}{cc|c} 0-\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}\right) & 1 & 0 \\ 1 & -1-\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}\right) & 0 \end{array}\right]$$

faith! $\rightarrow$ $$\left[\begin{array}{cc|c} \frac{1}{2}+\frac{\sqrt{5}}{2} & 1 & 0 \\ \ell\ell\ell & \ell\ell\ell & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} A & B & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \text{Get} \quad \vec{a} = \begin{pmatrix} B \\ -A \end{pmatrix}$$

$$\vec{a} = \begin{pmatrix} 1 \\ -\frac{1}{2} - \frac{\sqrt{5}}{2} \end{pmatrix} \quad \text{or better:} \quad \vec{a} = \begin{pmatrix} 2 \\ -1-\sqrt{5} \end{pmatrix}$$

For $r = -\frac{1}{2} + \frac{\sqrt{5}}{2}$ Get $\vec{a} = \begin{pmatrix} 2 \\ -1+\sqrt{5} \end{pmatrix}$

$$\vec{x} = c_1 \begin{pmatrix} 2 \\ -1-\sqrt{5} \end{pmatrix} e^{\overbrace{\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}\right)}^{r<0}t} + c_2 \begin{pmatrix} 2 \\ -1+\sqrt{5} \end{pmatrix} e^{\overbrace{\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)}^{r>0}t}$$

Saddle pt. unstable