Lesson 35 Predator-prey eqns

HWK 13 due Friday

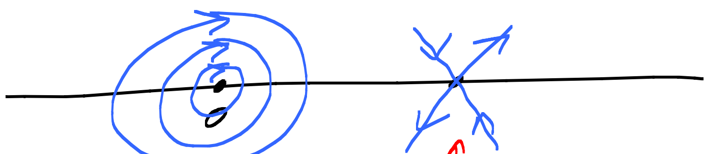
Read 9.5 on pred-prey.

Damped pendulum:

e.vals complex
Real part < 0
Asymp. stable spiral
Same for non-linear sys

e.vals, real, $\pm$
saddle pt., unstable
Same for non-linear sys.

Undamped case:



Center!
e.vals pure imag.

Saddle.
Same for non-linear sys

Delicate case! Can't conclude that non-linear sys has a center too at crt pt.
[Could be spiral in, out, or a center.]

Fancier analysis shows it is a center too.

Predator-prey eqns:

$$\begin{cases} \dfrac{dx}{dt} = \left(1 - \frac{1}{2}y\right)x & x \text{ moose} \\[2ex] \dfrac{dy}{dt} = \left(-\frac{3}{4} + \frac{1}{4}x\right)y & y \text{ wolves} \end{cases}$$

Crt pts: (A): $x\left(1 - \frac{1}{2}y\right) = 0 \leftarrow x = 0$ or $y = 2$

(B): $\left(-\frac{3}{4} + \frac{1}{4}x\right)y = 0$

$x = 0$ case: (B): $\left(-\frac{3}{4}\right) y = 0 \leftarrow y = 0$

$(0,0)$ is a crt pt.

$\underline{y = 2 \text{ case}}$: (B): $\left(-\frac{3}{4} + \frac{1}{4}x\right) \cdot \underline{\underline{2}} = 0 \leftarrow x = 3$

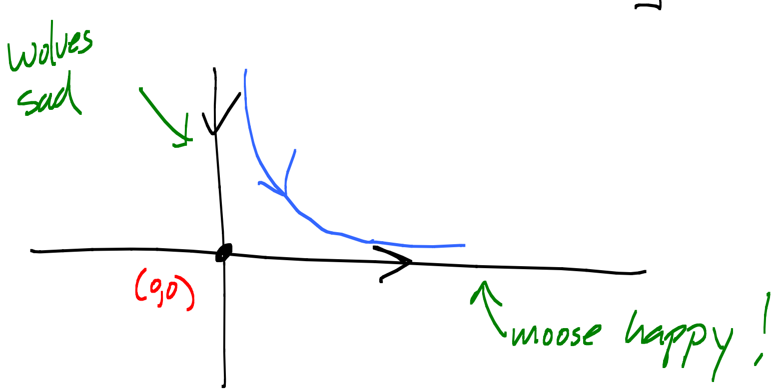
$(3,2)$ is a crt pt.

$\underline{\text{Linearize at crt pts}}$:
$$\begin{cases} \dfrac{dx}{dt} = x - \dfrac{1}{2}xy & = f(x,y) \\[2ex] \dfrac{dy}{dt} = -\dfrac{3}{4}y + \dfrac{1}{4}xy & = g(x,y) \end{cases}$$

$$\vec{x}' \doteq A\vec{x} \quad \text{where} \quad A = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}\Bigg|_{(x_0,y_0)} = \begin{bmatrix} 1-\frac{1}{2}y & -\frac{1}{2}x \\ \frac{1}{4}y & -\frac{3}{4}+\frac{1}{4}x \end{bmatrix}$$

$\underline{\text{At } (0,0)}$: $A = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{3}{4} \end{bmatrix}$ $\quad r = 1, -\frac{3}{4}$

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3/4\, t}$$



saddle pt.
unstable

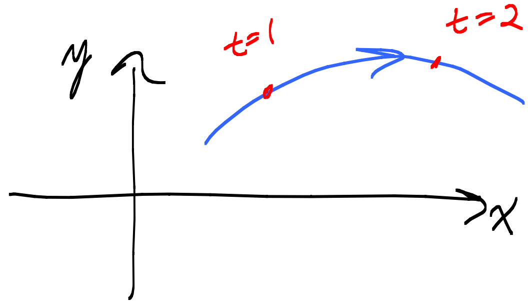
$\underline{\text{At } (3,2)}$: $A = \begin{bmatrix} 0 & -\frac{3}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$ $\quad r = \pm \frac{\sqrt{3}}{2} i$

center!

Darn! Delicate case.

Fancier analysis: Phase plane $x = F(t)$ (A)

$y = G(t)$ (B)



Idea: Eliminate t: (A): $t = F^{-1}(x)$

(B): $\underline{\underline{y = G(F^{-1}(x))}}$

Chain rule shows: $\dfrac{dy}{dx} = \dfrac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \dfrac{\left(-\frac{3}{4} + \frac{1}{4}x\right)y}{\left(1 - \frac{1}{2}y\right)x}$ Separable!

$$\int \frac{1 - \frac{1}{2}y}{y}\, dy = \int \frac{\left(-\frac{3}{4} + \frac{1}{4}x\right)}{x}\, dx$$

$$\int \frac{1}{y} - \frac{1}{2}\, dy = \int -\frac{3}{4}\frac{1}{x} + \frac{1}{4}\, dx$$

$$\ln y - \frac{1}{2}y = -\frac{3}{4}\ln x + \frac{1}{4}x + C$$

<u>Ouch!</u> Can't solve for x or y. But <u>can</u> show these define <u>closed</u> curves. So $(3,2)$ <u>is</u> a center.

Laplace transform

Read Chap. 6.

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt = F(s) \leftarrow \text{for big } s$$

Key properties: 1) It's a linear operator

2) $\mathcal{L}[f'(t)] = \underline{sF(s) - f(0)}$

$$\mathcal{L}[f''(t)] = s\, \underbrace{\mathcal{L}[f']}_{sF - f(0)} - f'(0)$$

$$= \underline{s^2 F(s) - f(0)s - f'(0)}$$

Aha! $\mathcal{L}$: ODE's $\longrightarrow$ algebra eqns

solve for $F(s)$!

Key property 3. Laplace transform can be undone!

Fact: If $\mathcal{L}[f_1] = \mathcal{L}[f_2]$ for $s > M$,

then $f_1 = f_2$. (assuming f's are piecewise smooth, etc.)

Ex: $y'' + 3y' + 4y = 0$ $\quad y(0) = 5$

$y'(0) = 7$

$$\mathcal{L}[\text{ODE}]: \quad \left(s^2 Y(s) - 5s - 7\right) + 3\left(sY(s) - 5\right) + 4Y(s) = 0$$

$$(s^2 + 3s + 4)\,Y = 5s + 7 + 15 = 5s + 22$$

$$Y(s) = \frac{5s+22}{s^2+3s+4}$$

Prob: What $y(t)$ has this as its L. Transf ?

Why do this? Particularly nice at handling piecewise defined forcing fcns.

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