Lesson 36 Laplace trasform

No class on Monday (Video instead)

$$\mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st}\, dt = F(s) \leftarrow \text{fcn of } s, \quad s > M$$

fcn of t, $t \geq 0$

EX: $\mathcal{L}[e^{at}] = \int_0^\infty e^{at} e^{-st}\, dt$

$$= \int_0^\infty e^{(a-s)t}\, dt = \lim_{B\to\infty} \left[\frac{1}{a-s} e^{(a-s)t} \right]_0^B$$

need $a-s<0$, $s>a$

$$= 0 - \frac{1}{a-s} e^{(a-s)\cdot 0} \quad \text{if } s>a$$

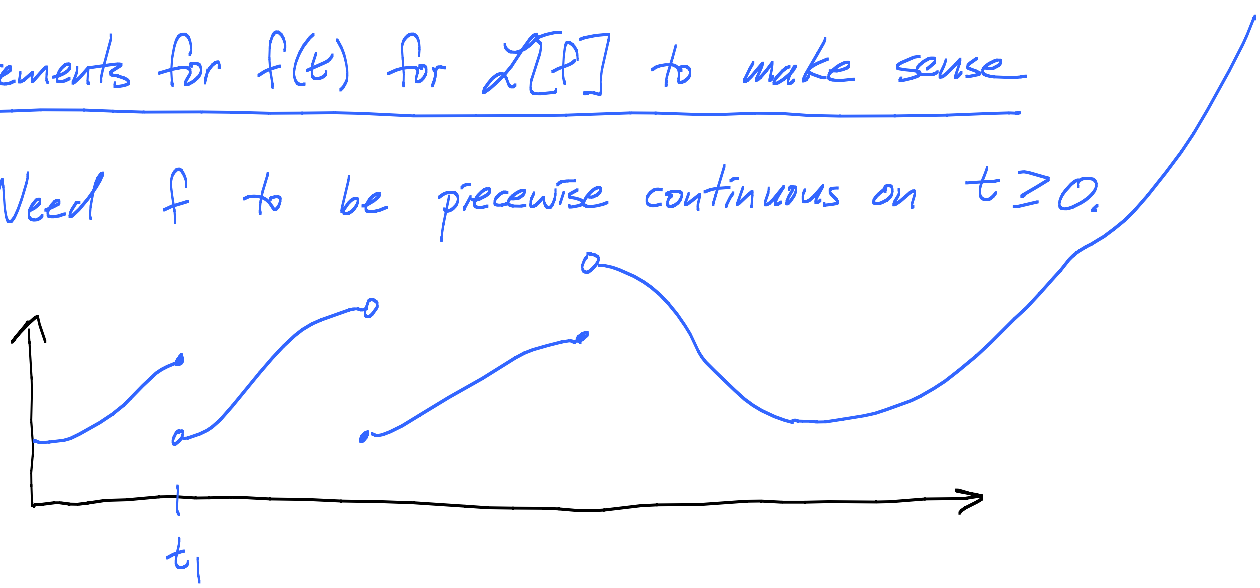
$$= \frac{1}{s-a}$$

Laplace Transform table:

$f(t)$	$F(s)$
e^{at}	$\frac{1}{s-a}$, $s>a$

Requirements for $f(t)$ for $\mathcal{L}[f]$ to make sense

1) Need f to be piecewise continuous on $t \geq 0$.



2) f needs to be of "exponential type" meaning there are constants M and K such that

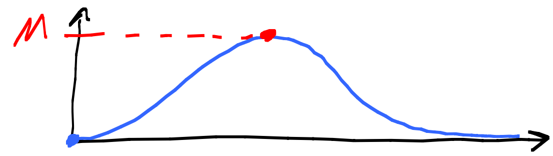
$$|f(t)| \le M e^{kt} \quad \text{for} \quad t \ge 0.$$

Why: $\displaystyle\int_0^\infty \underbrace{f(t)\, e^{-st}} \, dt$

$$|f(t) e^{-st}| \le M e^{kt} e^{-st} = M e^{(k-s)t}$$ ← need $s > k$

EX: $t^2 e^{3t} \cos 2t$ is of exp type:

$\dfrac{t^2}{e^t} \longrightarrow 0$ as $t \to \infty$.



Let $M = \max\limits_{t \ge 0} t^2/e^t$.

$\left|\dfrac{t^2}{e^t}\right| \le M$. So $t^2 \le M e^t$. [t^2 is of exp type]

Now $\left|t^2 e^{3t} \cos 2t\right| \le (M e^t) e^{3t} \cdot 1 \le M e^{4t}$, $t \ge 0$.

EX: $e^{(e^t)}$ is not of exp type!

The s-shifting rule: $\mathcal{L}[f(t)] = F(s)$

$$\mathcal{L}[e^{at} f(t)] = F(s-a)$$ ← shift by a

Why: $\mathcal{L}[e^{at} f(t)] = \displaystyle\int_0^\infty e^{at} f(t)\, e^{-st} \, dt$

$$= \int_0^\infty e^{(a-s)t} f(t)\, dt$$

$$= \int_0^\infty e^{-\sigma t} f(t)\, dt = F(\sigma) \qquad \sigma = -(a-s) = s-a \quad \checkmark$$

Table:

$f(t)$	$F(s)$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$

Shifted	
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos bt$	$\frac{(s-a)}{\underbrace{(s-a)^2+b^2}_{\text{completing square!}}}$

Fact: $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

(+): $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ $\qquad$ (−): $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

Famous formulas: $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ $\qquad$ $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

Completing the square: $As^2 + Bs + C$

Step 1: Factor out A: $\quad A(s^2 + bs + c)$

where $b = \frac{B}{A}$, $c = \frac{C}{A}$.

Step 2: $s^2 + bs + c = \left(s + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)$

$\frac{1}{2}$ times b

clean up term

Prob: Find $\mathcal{L}^{-1}\left[\frac{2s}{\underbrace{s^2-4s+13}}\right]$ ← complex roots

$$\frac{2s}{(s-2)^2+9}$$ ← wish we had $s-2$ here

$$\frac{2\left[(s-2)+2\right]}{(s-2)^2+3^2}$$

$$2\cdot\frac{s-2}{(s-2)^2+3^2}+\frac{4}{3}\cdot\frac{3}{(s-2)^2+3^2}$$

$$=\mathcal{L}\left[\underbrace{2e^{2t}\cos 3t+\frac{4}{3}e^{2t}\sin 3t}_{\text{ans}}\right]$$

EX:

$$\frac{s+4}{s^2+4s+3} \leftarrow \text{real roots} = \frac{s+4}{(s+1)(s+3)}$$

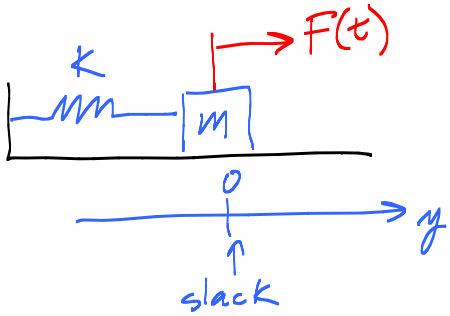
$$=\frac{A}{s+1}+\frac{B}{s+3}$$ ← Do partial fractions

$$\mathcal{L}^{-1}=Ae^{-t}+Be^{-3t}$$

Note: $\mathcal{L}\left[e^{at}\right]=\frac{1}{s-a}$, $\mathcal{L}\left[e^{-at}\right]=\frac{1}{s-(-a)}=\frac{1}{s+a}$

Very cool thing about $\mathcal{L}$: Great at discontinuous $f(t)$'s.

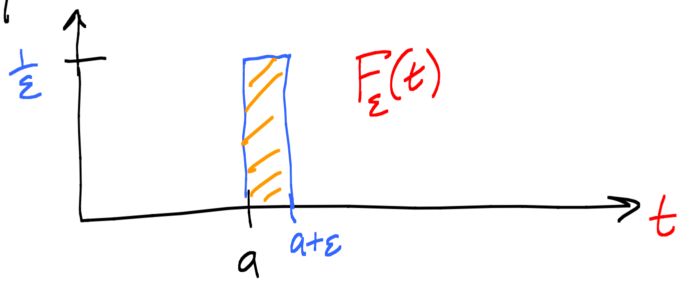
Prob:



$$m\frac{d^2y}{dt} + ky = F(t)$$

$$y(0) = y_0$$
$$y'(0) = y_0'$$

Impulse fcn: Hit it with a hammer!



Area = 1

$$\mathcal{L}[F_\varepsilon(t)] = \int_0^\infty F_\varepsilon(t)\, e^{-st}\, dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\, e^{-st}\, dt$$

$$\longrightarrow e^{-as} \leftarrow \mathcal{L}[F] \leftarrow \text{ideal sledge hammer}$$

Cool thing

$$my'' + ky = F_\varepsilon$$

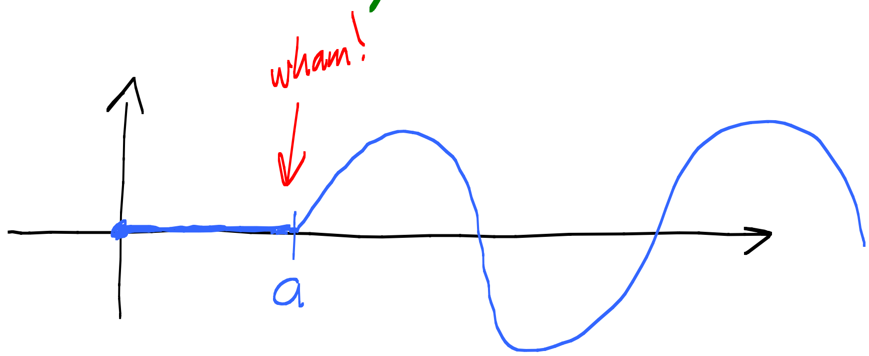
$$m[s^2 Y(s) - \underbrace{sy_0 - y_0'}_{\text{take these } = 0}] + kY = \mathcal{L}[F_\varepsilon]$$

Let $\varepsilon \to 0$

$$[ms^2 + k]\, Y = e^{-as}$$

$$Y(s) = \frac{e^{-as}}{ms^2 + k}$$

Cool thing: $\mathcal{L}^{-1}$ of $\dfrac{e^{-as}}{ms^2+k}$ is a perfecty good fcn of t and gives sol^n to perfect hit with a sledge hammer (unit impulse $= \int F(t)\,dt$).



Impulse & momentum:

Impulse = Δ momentum

$$\int F(t)\,dt = \Delta mv$$