Lesson 37 Using Laplace Transform

Read 6.1 – 6.3

$$\mathcal{L}[\underbrace{f(t)}_{\text{function on } t\geq 0}] = \int_0^\infty e^{-st} f(t)\,dt = F(s) \leftarrow \text{new function of } s \text{ defined for } s>M \text{ for some } M.$$

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad \text{for } s>a.$$

Table of Laplace Transforms

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$f(t)$	$F(s)$	
e^{at}	$\frac{1}{s-a}$	$(s>a)$
$\sin bt$	$\frac{b}{s^2+b^2}$	$(s>0)$
$\cos bt$	$\frac{s}{s^2+b^2}$	$(s>0)$
1	$1/s$	$(s>0)$
t	$1/s^2$	$(s>0)$
t^n	$\frac{n!}{s^{n+1}}$	$(s>0)$
$\sinh bt$	$\frac{b}{s^2-b^2}$	$(s>b)$
$\cosh bt$	$\frac{s}{s^2-b^2}$	$(s>b)$

critical that $s>0$!

EX: $$\mathcal{L}[1] = \int_0^\infty 1\cdot e^{-st}\,dt = \lim_{B\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^B$$

$$= 0 - \left(-\frac{1}{s}\right) = 1/s \checkmark$$

EX: $$\mathcal{L}[t^2] = \int_0^\infty \underbrace{t^2}_{u}\underbrace{e^{-st}\,dt}_{dv}$$

$u = t^2 \qquad du = 2t\,dt$

$dv = e^{-st}dt \qquad v = -\frac{1}{s}e^{-st}$

$$\int u\,dv = uv - \int v\,du = t^2\left(-\tfrac{1}{s}e^{-st}\right) - \int\left(-\tfrac{1}{s}e^{-st}\right)(2t\,dt)$$

$$= -t^2\frac{1}{s}e^{-st} + \frac{2}{s}\underbrace{\int te^{-st}\,dt}_{\text{integrate by parts again. } u=t.}$$

$s>0$ critical here.

Note: The Lap. T. is a linear operator:

$$\mathcal{L}[c_1f_1 + c_2f_2] = c_1F_1 + c_2F_2$$

Fun thing: $\sinh bt = \frac{1}{2}\left(e^{bt} - e^{-bt}\right)$

So $\mathcal{L}[\sinh bt] = \frac{1}{2}\cdot\left[\frac{1}{s-b} - \frac{1}{s-(-b)}\right]$ $(s>b)$

$$= \frac{1}{2}\cdot\frac{2b}{s^2-b^2} \checkmark$$

EX: $\mathcal{L}[\sin bt] = \int_0^\infty e^{-st}\sin bt\,dt$ ← Integrate by parts twice. Solve for $\int$.

More fun: $\mathcal{L}[\sin bt] = \mathcal{L}\left[\frac{1}{2i}\left(e^{ibt} - e^{-ibt}\right)\right]$

$$= \frac{1}{2i}\left[\frac{1}{s-ib} - \frac{1}{s-(-ib)}\right] = \frac{b}{s^2+b^2} \;!$$

Huge fact #1: $\mathcal{L}[f'(t)] = sF(s) - f(0)$

Why: $\mathcal{L}[f'(t)] = \int_0^\infty f'(t)e^{-st}\,dt$

$$\int \underbrace{e^{-st}}_{u} \underbrace{f'(t)\,dt}_{dv} \qquad u = e^{-st} \qquad du = -se^{-st}\,dt$$

$$dv = f'(t)\,dt \qquad v = \int f'(t)\,dt = \underline{\underline{f(t)}}$$

$$= uv - \int v\,du = e^{-st} f(t) - \int f(t)\,(-se^{-st}\,dt)$$

f Expo Type : $|f(t)| \le Me^{kt}$

So $e^{-st} f(t) \to 0$ as $t \to \infty$ provided that $\underline{\underline{s > k}}$.

$$\text{So} \quad \underbrace{\int_0^\infty e^{-st} f'(t)\,dt}_{\mathcal{L}[f']} = \underbrace{\left[0 - e^{-s\cdot 0} f(0)\right]}_{-f(0)} + s \underbrace{\int_0^\infty f(t) e^{-st}\,dt}_{F(s)} \checkmark$$

<u>Repeat huge fact</u>:

$$\mathcal{L}[f] = F$$

$$\mathcal{L}[f'] = sF(s) - f(0)$$

$$\mathcal{L}[(f')'] = s\underbrace{\mathcal{L}[f']}_{sF(s)-f(0)} - f'(0) = s^2F(s) - sf(0) - f'(0)$$

$$\mathcal{L}[f^{(n)}(t)] = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

<u>EX</u>:

$$y'' + 5y' + 6y = 0 \qquad \begin{cases} y(0) = 3 \\ y'(0) = 5 \end{cases}$$

$$\mathcal{L}[y(t)] = Y(s)$$

$$\mathcal{L}[y'(t)] = sY(s) - \underbrace{y(0)}_{3}$$

$$\mathcal{L}[y''(t)] = s^2 Y(s) - s\underbrace{y(0)}_{3} - \underbrace{y'(0)}_{5}$$

Hit ODE with $\mathcal{L}$: $\quad \mathcal{L}[y'' + 5y' + 6y] = \mathcal{L}[0] = 0$

$$\mathcal{L}[y''] + 5\mathcal{L}[y'] + 6\mathcal{L}[y] = 0$$

Algebra! $\rightarrow \left(s^2 Y - 3s - 5\right) + 5\left(sY - 3\right) + 6Y = 0$

$$(s^2+5s+6)Y = 3s+20$$

Partial Fractions!

$$Y = \frac{3s+20}{s^2+5s+6} = \frac{3s+20}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

Mult. by denom $(s+2)(s+3)$: $\quad 3s+20 = A(s+3) + B(s+2)$

$$= \underbrace{(A+B)}_{3}s + \underbrace{(3A+2B)}_{20}$$

$$\begin{cases} A+B = 3 \\ 3A+2B = 20 \end{cases}$$

Cramer's Rule: $\quad A = \dfrac{\det\begin{bmatrix} 3 & 1 \\ 20 & 2 \end{bmatrix}}{\det\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}} = \dfrac{-14}{(-1)} = 14$

$$B = \frac{\det\begin{bmatrix} 1 & 3 \\ 3 & 20 \end{bmatrix}}{\det\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}} = \frac{11}{(-1)} = -11$$

So $\quad Y = \dfrac{14}{s+2} - \dfrac{11}{s+3}$

$\frac{1}{s+2} = \mathcal{L}[e^{-2t}] \qquad \frac{1}{s+3} = \mathcal{L}[e^{-3t}]$

Read off: $\quad \boxed{y = 14e^{-2t} - 11e^{-3t}}$

Another way to think: $\quad y = \mathcal{L}^{-1}[Y] = 14\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - 11\mathcal{L}^{-1}\left[\frac{1}{s+3}\right]$

Dealing with $\dfrac{As+B}{as^2+bs+c}$ $\begin{array}{l} \leftarrow \text{from initial conditions} \\ \leftarrow \text{characteristic poly} \end{array}$

Cases: 1) $as^2+bs+c = a(s-r_1)(s-r_2)$ $\leftarrow$ partial fractions

2) $as^2+bs+c = a(s-r)^2$

3) as^2+bs+c has complex roots. Complete the square!

Completing square: $as^2+bs+c = a\left(s^2+\underbrace{\frac{b}{a}}_{B}s+\underbrace{\frac{c}{a}}_{C}\right)$

$$s^2+Bs+C = \left(s+\frac{B}{2}\right)^2+\left(C-\frac{B^2}{4}\right)$$

$\frac{1}{2}B$