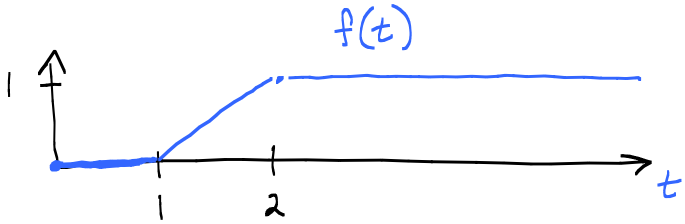
Final Exam
Mon 1–3 pm
BRWN 1154

Lesson 38 Laplace transform of piecewise defined functions

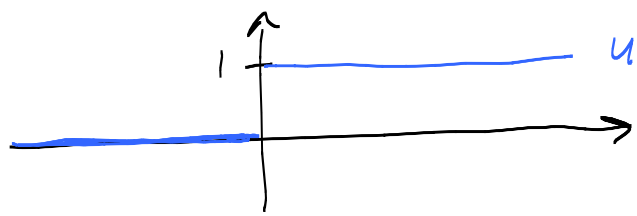
EX:



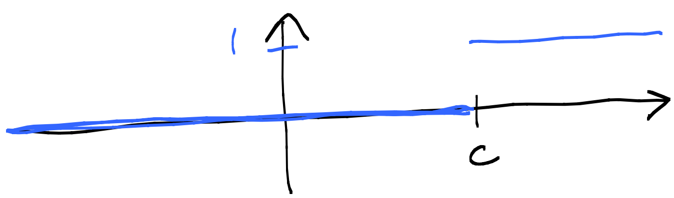
$$f(t) = \begin{cases} 0 & 0 \le t \le 1 \\ t-1 & 1 \le t \le 2 \\ 1 & t \ge 2 \end{cases}$$

$$\mathcal{L}[f] = \int_0^\infty f(t)e^{-st}\,dt = 0 + \int_1^2 (t-1)e^{-st}\,dt + \int_2^\infty 1\cdot e^{-st}\,dt$$

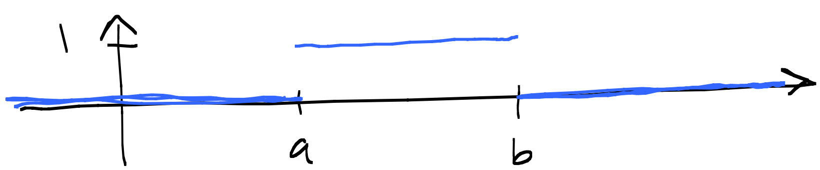
Def^n: Heaviside function (unit step function) $u(t)$



$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

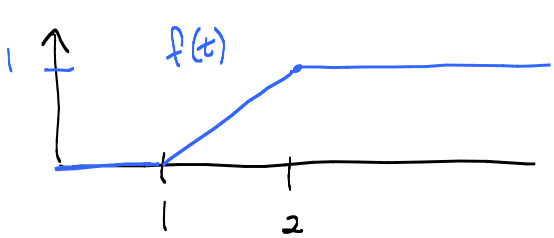


$$u_c(t) = u(t-c)$$



$$u_a(t) - u_b(t)$$

on at $t=a$ off at $t=b$



$$f(t) = 0\cdot\left[u(t) - u_1(t)\right] + (t-1)\left[u_1(t) - u_2(t)\right] + 1\cdot u_2(t)$$

Two big facts: 1) s-shifting $\mathcal{L}[f(t)] = F(s)$

$$\mathcal{L}[e^{at} f(t)] = F(s-a)$$

EX: $\mathcal{L}[\sin bt] = \dfrac{b}{s^2+b^2}$

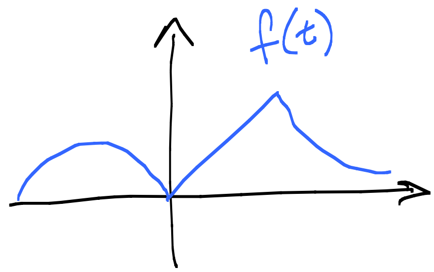
$$\mathcal{L}[e^{at} \sin bt] = \frac{b}{(s-a)^2 + b^2}$$

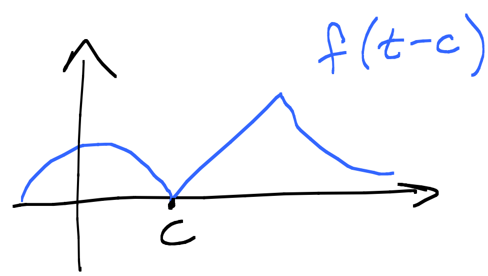
2) t-shifting rule: $\mathcal{L}[u_c(t) f(t-c)] = e^{-cs} F(s)$

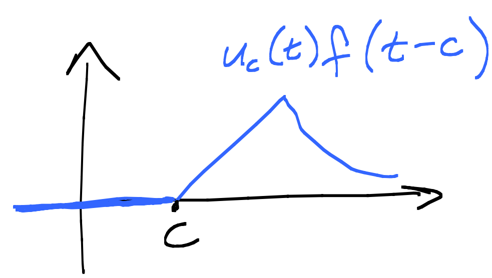
EX: Take $f(t) \equiv 1$ in (2). $\mathcal{L}[1] = \frac{1}{s}$

$$\mathcal{L}[u_c(t)] = \mathcal{L}[u_c(t) \cdot 1] = e^{-cs}\frac{1}{s} = \frac{e^{-cs}}{s}$$

Why 2:







$$\mathcal{L}[u_c(t) f(t-c)] = \int_c^\infty f(\underbrace{t-c}_{\tau})\, e^{-st}\, \underbrace{dt}_{d\tau}$$

$\tau = t-c$

so

$t = \tau + c$

When $t=c$: $\tilde{\tau} = c-c = 0$

$t \to \infty$: $\tilde{\tau} \to \infty$

$$= \int_{\tilde{\tau}=0}^{\infty} f(\tilde{\tau})\, \underbrace{e^{-s(\tilde{\tau}+c)}}_{e^{-s\tilde{\tau}} e^{-cs}}\, d\tilde{\tau}$$

$$= e^{-cs} \underbrace{\int_0^{\infty} f(\tilde{\tau})\, e^{-s\tilde{\tau}}\, d\tilde{\tau}}_{F(s)}$$

EX: Find $\mathcal{L}[u_3(t)\, t^2]$

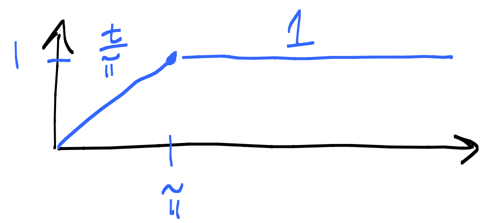
(2) $\mathcal{L}[u_c(t) f(t-c)] = e^{-cs} F(s)$

$$t^2 = \big((t-3)+3\big)^2 = \underbrace{(t-3)^2 + 6(t-3) + 9}_{f(t-3)}$$

So $f(t) = t^2 + 6t + 9$

$F(s) = \frac{2}{s^3} + 6 \cdot \frac{1}{s^2} + \frac{9}{s}$

So $\mathcal{L}[u_3(t)\, t^2] = e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)$

Prob: $y'' + y =$ 

$g(t)$ $\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$

$$\Big(s^2 Y - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0}\Big) + Y = G(s)$$

$$Y = \underbrace{\frac{1}{s^2+1}}_{\text{"Transfer fcn"}}\, G(s)$$

$$g(t) = \frac{t}{\pi}\left[\underset{\substack{\uparrow\\ \text{on at}\\ t=0}}{u(t)} - \underset{\substack{\uparrow\\ \text{off at}\\ t=\pi}}{u_\pi(t)}\right] + 1 \cdot \underset{\substack{\uparrow\\ \text{on at}\\ t=\pi}}{u_\pi(t)}$$

(never off)

$$g(t) = \underbrace{u_0(t)\frac{t-0}{\pi}}_{\text{same as } \frac{t}{\pi}} - u_\pi(t)\frac{(t-\pi)+\pi}{\pi} + u_\pi(t)$$

So $$G(s) = \frac{1}{\pi}\left(\frac{1}{s^2} + e^{-\pi s}\cdot\frac{1}{s^2}\right) = \frac{1}{\pi}\frac{1}{s^2}\left(1 + e^{-\pi s}\right)$$

$$Y(s) = \underbrace{\frac{1}{s^2(s^2+1)}}_{H(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs+D}{s^2+1}}\frac{1}{\pi}\left(1 + e^{-\pi s}\right)$$

(Partial fractions)

$$1 = A(s^2+1) + Bs(s^2+1) + (Cs+D)s^2$$

$A=1$
$B=0$
$C=0$
$D=-1$

$$1 = \underbrace{(B+C)}_{0}s^3 + \underbrace{(A+D)}_{0}s^2 + \underset{\substack{\uparrow\\0}}{B}s + \underset{\substack{\uparrow\\1}}{A}$$

$$\boxed{H(s) = \frac{1}{s^2} - \frac{1}{s^2+1}}$$

$$h(t) = t - \sin t$$

$$Y = \frac{1}{\pi}H(s)\left[1 - e^{-\pi s}\right] = \frac{1}{\pi}H(s) - \frac{1}{\pi}e^{-\pi s}H(s)$$

Finally $$y(t) = \frac{1}{\pi}h(t) - \frac{1}{\pi}u_\pi(t)h(t-\pi)$$

$$\mathcal{L}[u_c(t)f(t-c)] = e^{-cs}F(s)$$

$$y(t) = \frac{1}{\pi}(t - \sin t) - \frac{1}{\pi} u_\pi(t)\left[(t-\pi) - \underbrace{\sin(t-\pi)}\right]$$

$\sin(t-\pi) = -\sin t$

$$= \begin{cases} \frac{1}{\pi}(t - \sin t) & 0 \le t \le \pi \\ 1 - \frac{2}{\pi}\sin t & t \ge \pi \end{cases}$$

Question: Is $\mathcal{L}[f(t)g(t)] = F(s)\,G(s)$?

No, no, no!

But $\mathcal{L}[(f*g)(t)] = F(s)\,G(s)$

f convolved with g

$$(f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

Prob: $y'' + y = g(t)$ $\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$

$$Y = \underbrace{\frac{1}{s^2+1}}_{\mathcal{L}[\sin t]} G(s)$$

Sol^n: $y(t) = \sin t * g(t) = \int_0^t \sin\tau \; g(t-\tau)\,d\tau$

Laplace transform: 6.1 – 6.5