Review 1

Separable eqns : $\frac{dy}{dx} = F(x)\, G(y)$

$$\int \frac{dy}{G(y)} = \int F(x)\, dx \leftarrow$$

don't forget + C (on right side)

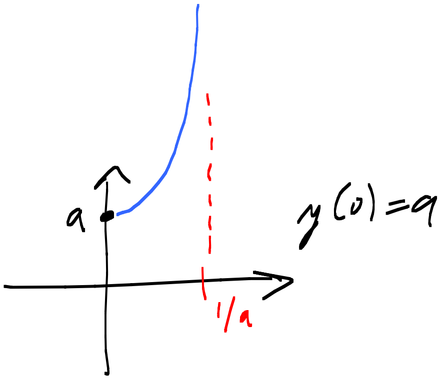
Solve for y.

EX: $\frac{dy}{dx} = y^2$

$$\int \frac{dy}{y^2} = \int dx$$

$$-\frac{1}{y} = x + C$$

$$y = \frac{-1}{x+C}$$



$$a = \frac{-1}{0+C} \qquad C = -\frac{1}{a} \qquad y = \frac{-1}{x - \frac{1}{a}}$$

Hmmm. What if $a=0$? E & U Thm says a solⁿ to $y' = y^2$ with $y(0)=0$ exists on some interval $(-\varepsilon, \varepsilon)$.

$y \equiv 0$ is the missing solⁿ! Aha! $y \equiv 0$ is $\lim_{c \to \infty} \frac{-1}{x+c}$.

Linear eqns: $A(x)\frac{dy}{dx} + B(x)\,y = C(x)$

1) Divide by $A(x)$: Get

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

2) $u = e^{\int P(x)\,dx}$

$$\underbrace{u\frac{dy}{dx} + uPy}_{\frac{d}{dx}\left[uy\right]} = uQ$$

$$uy = \int uQ\,dx + C$$

$$y = \frac{1}{u}\int uQ\,dx + \frac{C}{u}$$

Ex: $\boxed{x^2}\frac{dy}{dx} + 2xy = 3x$

zeroes are bad news

1) $\frac{dy}{dx} + \boxed{\frac{2}{x}}\,y = \frac{3}{x}$ ← mult by u

bad at $x=0$

2) $u = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{\ln|x|^2} = |x|^2 = x^2$

$$\underbrace{x^2\left(\frac{dy}{dx} + \frac{2}{x}\,y\right)}_{\frac{d}{dx}\left[x^2 y\right]} = \frac{3}{x}\cdot x^2$$

$$\frac{d}{dx}\left[x^2 y\right] = 3x$$

$$x^2 y = \int 3x \, dx = \frac{3}{2}x^2 + C$$

$$\boxed{y = \frac{3}{2} + \frac{C}{x^2}} \leftarrow$$ ouch! at $x=0$

E & U Thm: P, Q cont. on (a, b). Then sol'ns exist and are good on (a, b). [$\frac{2}{x}$ blows up at $x=0$.]

Advice: Use linear method over others when you can.

Prob:

$$\frac{dy}{dx} = \frac{x^2+y^2}{xy} = \frac{x}{y} + \frac{y}{x} = \underbrace{\frac{1}{\left(\frac{y}{x}\right)} + \left(\frac{y}{x}\right)}_{F\left(\frac{y}{x}\right)}$$

Homogeneous: Let $\boxed{v = \frac{y}{x}}$

solve for y → $y = xv$

$$\frac{dy}{dx} = 1 \cdot v + x \cdot \frac{dv}{dx}$$

$$\underbrace{\frac{dy}{dx}} = F\left(\frac{y}{x}\right)$$

$$v + x\frac{dv}{dx} = F(v) \leftarrow$$ seperable!

$$v + x\frac{dv}{dx} = \frac{1}{v} + v$$

$$\int v \, dv = \int \frac{dx}{x}$$

$$\frac{1}{2}v^2 = \ln|x| + C$$

$$v^2 = 2\ln|x| + 2C$$

$$\frac{y}{x} = v = \sqrt{2\ln|x| + 2C}$$

$$\boxed{y = x\sqrt{2\ln|x| + K}}$$

$K = 2C$

Exact eqns: $\underbrace{M}_{\frac{\partial \varphi}{\partial x}} + \underbrace{N}_{\frac{\partial \varphi}{\partial y}} \frac{dy}{dx} = 0$ $\qquad M\,dx + N\,dy = 0$

$$\frac{d}{dx}\left(\varphi(x,y)\right) = 0 \qquad \varphi(x,y) = C$$

Need $\frac{\partial}{\partial x}\left[\begin{matrix}\text{Thing by}\\ dy\end{matrix}\right] = \frac{\partial}{\partial y}\left[\begin{matrix}\text{Thing by}\\ dx\end{matrix}\right]$

$$N_x = M_y$$

EX: $\underbrace{\left(2xy + 2xy^2 + 1\right)}_{M} + \underbrace{\left(x^2 + 2x^2y + 2y\right)}_{N}\frac{dy}{dx} = 0$

$y(1) = 2$

$$N_x = 2x + 4xy + 0 \overset{?}{=} 2x + 4xy + 0 = M_y$$

✓ yes! Exact!

Can find $\varphi(x,y)$ such that $\frac{\partial \varphi}{\partial x} = M$, $\frac{\partial \varphi}{\partial y} = N$

Want: $$\frac{\partial \varphi}{\partial x} = 2xy + 2xy^2 + 1 \qquad (A)$$

$$\frac{\partial \varphi}{\partial y} = x^2 + 2x^2y + 2y \qquad (B)$$

Use (A): $$\varphi = \int 2xy + 2xy^2 + 1 \, dx + \underline{\underline{C(y)}}$$

$$\boxed{\varphi = x^2y + x^2y^2 + x + C(y)}$$

Now use (B): $$\frac{\partial}{\partial y}\Big(\underbrace{x^2y + x^2y^2 + x + C(y)}_{\varphi}\Big) = \underbrace{x^2 + 2x^2y + 2y}_{N}$$

$$x^2 + 2x^2y + C'(y) = x^2 + 2x^2y + 2y$$

$$\boxed{C'(y) = 2y}$$

So $C(y) = \int 2y \, dy = y^2$ ← skip +K here.

Finally $\varphi(x,y) = x^2y + x^2y^2 + y^2 = c$

IVP: $y(1)=2$ $\qquad 1^2 \cdot 2 + 1^2 \cdot 2^2 + 2^2 = c$

$$10 = C$$

$$(1+x^2)y^2 + x^2y - 10 = 0$$

$$y = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$ ← pick one with $y(1)=2$.

Reduction of order: $(x-1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x+1)y = 0$

Given that $y_1(x) = e^x$ solves eqn, find gen^l sol^n.

$$y_2(x) = u(x)\,y_1(x)$$

Formula for u: Standard form! $\frac{d^2y}{dx^2} \underline{- \frac{2x}{x-1}} \frac{dy}{dx} + \frac{x+1}{x-1}y = 0$

$$-\frac{2x}{x-1} = P(x)$$

$$\boxed{u' = \frac{1}{y_1^2} e^{-\int P(x)\,dx}}$$

$$u' = \frac{1}{(e^x)^2} e^{-\int \frac{-2x}{x-1}\,dx}$$

$$\begin{array}{r} 2 \\ x-1\,\overline{)\,2x} \\ \underline{2x-2} \\ 2 \end{array}$$

$$\frac{2x}{x-1} = 2 + \frac{2}{x-1}$$

$$u' = e^{-2x} e^{\int \frac{2[(x-1)+1]}{x-1}dx}$$

$$= e^{-2x} e^{\int 2 + \frac{2}{x-1}\,dx} = e^{-2x} e^{2x + 2\ln|x-1|}$$

$$u' = (x-1)^2$$

So $u = \int (x-1)^2\,dx = \frac{1}{3}(x-1)^3 \leftarrow$ no $+C$

$$y_2 = u y_1 = \frac{1}{3}(x-1)^3 e^x$$

Gen^l sol^n $y = c_1 e^x + c_2 (x-1)^3 e^x \leftarrow$ absorb $\frac{1}{3}$ into c_2

Prob: $y'' + 5y' + 6y = xe^{-2x}$

Homog eqn: $r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r = -2, -3$

Homog sol^n $y_c = c_1 e^{-2x} + c_2 e^{-3x}$

Table:

$$y_p = (Ax + B)e^{-2x}$$

$$= Axe^{-2x} + \underbrace{Be^{-2x}}$$

Ouch! solves homog

$$\boxed{y_p = x(Ax+B)e^{-2x}}$$

$$= \underset{\text{safe}}{Ax^2e^{-2x}} + \underset{\text{safe}}{Bxe^{-2x}}$$

Gen^l Sol^n: $y_c + y_p$