Review lecture 2

Final Exam Monday 1:00-3:00 pm in BRWN 1154

20 questions — part multiple choice, part short answer. Show work.
Laplace transform table on page 1. 1 crib sheet handwritten only, both sides.

Find the general solution of

$$y'' + 2y' + 2y = 5e^{-2t}\sin(t)$$

L.Transf: Add $\begin{cases} y(0) = 7 \\ y'(0) = 13 \end{cases}$ Find the L.T. $Y(s)$ of the sol^n $y(t)$ to IVP.

$$\mathcal{L}[y'] = sY(s) - y(0) = sY - 7$$

$$\mathcal{L}[y''] = s^2 Y(s) - sy(0) - y'(0) = s^2 Y - 7s - 13$$

$$(s^2 Y - 7s - 13) + 2(sY - 7) + 2Y = 5 \frac{1}{(s+2)^2 + 1^2}$$

Solve for $Y(s)$. Done!

Better:
$$r^2 + 2r + 2 = 0$$
$$(r+1)^2 + 1 = 0$$
$$(r+1) = \pm i$$
$$r = -1 \pm i$$

Homog sol^n

$$y_c = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$$

Use Undet Coeff to get y_p:

Try $y_p = Ae^{-2t}\cos t + Be^{-2t}\sin t$ ← safe! Do not solve homog

If RHS were $5e^{-t}\sin t$:

$$y_p = t\left(Ae^{-t}\cos t + Be^{-t}\sin t\right)$$ ← t^k, k smallest so that no single term solves homog

Given $A = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 4 & -2 & -3 \end{pmatrix}$ has eigenvalues -1, $-1+i$, $-1-i$ and corresponding eigenvector

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1+i \\ i \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1-i \\ -i \\ 2 \end{pmatrix}.$$

Find the **real** (meaning: does not involve complex numbers) general solution of $y' = Ay$.

$\vec{y}\,' = A\vec{y}$

$\vec{y}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-t}$

Plan: Get other two sol'ns from one complex sol'n

$$\begin{pmatrix} 1+i \\ i \\ 2 \end{pmatrix} \underbrace{e^{(-1+i)t}}_{\substack{e^{-t}e^{it} \\ e^{-t}\cos t + i e^{-t}\sin t}}$$

$$\left[\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + i\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right]\left(e^{-t}\cos t + i e^{-t}\sin t\right)$$

$$\underbrace{\left[\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^{-t}\cos t - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t}\sin t\right]}_{} + i\underbrace{\left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t}\cos t + \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^{-t}\sin t\right]}_{}$$

$i^2 = -1$

$$\vec{y}_2 = \begin{pmatrix} \cos t - \sin t \\ -\sin t \\ 2\cos t \end{pmatrix} e^{-t} \qquad \vec{y}_3 = \begin{pmatrix} \cos t + \sin t \\ \cos t \\ 2\sin t \end{pmatrix} e^{-t}$$

Gen'l sol'n $\quad c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 \leftarrow$ real valued sol'n!

Find all values of k for which the equilibrium point $(0,0)$ of the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \underbrace{\begin{pmatrix} -2 & k \\ -1 & 0 \end{pmatrix}}_{\mathbb{A}} \begin{pmatrix} x \\ y \end{pmatrix}$$

is a saddle point.

a) $k > 0$

b) $k > 1$

c) $k < 0$

d) $k < 1$

e) $k \geq 1$

$$\det(\mathbb{A} - r\mathbb{I})$$

$$= \det \begin{bmatrix} -2-r & k \\ -1 & 0-r \end{bmatrix}$$

$$= r^2 + 2r + k = 0$$

e. vals: $r = \dfrac{-2 \pm \sqrt{4-4k}}{2} = -1 \pm \sqrt{1-k}$

Cases: $1-k<0$ $\quad r = \underbrace{-1}_{} \pm \lambda i \quad \lambda = \sqrt{|1-k|}$ Spiral!

$Re<0$: Spiral's IN.

Stable. In fact Asymp. Stable.

Case: $1-k=0$. $\quad r=-1,-1 \quad \vec{x} = c_1 \vec{a} e^{-t} + c_2 (\vec{a} t e^{-t} + \vec{b} e^{-t})$

Asympt. Stable. Improper node.

Case: $-1 \pm \sqrt{1-k} \qquad 1-k>0$

Hmmm: Sub cases $0 < 1-k < 1$

$r_1 : -1 - \sqrt{1-k} < 0$

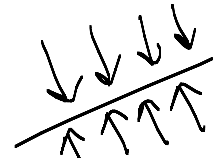
$r_2 : -1 + \underbrace{\sqrt{1-k}}_{<1} < 0$ too.

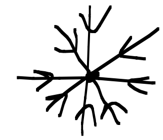
Asymp. Stable improper node

Sub case $1-k>1$. $r_1<0$. $r_2>0$. Saddle pt. Unstable.

Case $1-k=1$: $-1-1=-2$ ← weird case

$-1+1=0$





proper node

One of the critical points (equilibrium solutions) of

$$\frac{dx}{dt} = e^{-x+y} - \cos(x), = f(x,y)$$
$$\frac{dy}{dt} = \sin(x-3y) = g(x,y)$$

is $(0,0)$.

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -e^{-x}e^{y} + \sin x & e^{-x}e^{y} \\ \cos(x-3y)\cdot 1 & \cos(x-3y)\cdot(-3) \end{bmatrix}$$

a) Find the corresponding linear system near $(0,0)$.

$$\vec{x}\,' = A\vec{x}$$

where $A = J\Big|_{(x_0,y_0)} = \begin{bmatrix} -1 & 1 \\ 1 & -3 \end{bmatrix}$

↑ Crt. pt.

Laplace transform: Know how to find the inverse Laplace T. of

a) $$\frac{s+3}{s^2+4s+4} = \frac{s+3}{(s+2)^2} = \frac{[(s+2)-2]+3}{(s+2)^2} = \frac{1}{s+2} + \frac{1}{(s+2)^2}$$

$$= \mathcal{L}\left[e^{-2t} + te^{-2t}\right]$$

b) $$\frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+1)(s+2)} \underset{\text{partial fractions}}{=} \frac{A}{s+1} + \frac{B}{s+2}$$

$$= \mathcal{L}\left[Ae^{-t} + Be^{-2t}\right]$$

c) $$\frac{s+3}{s^2+2s+2} = \frac{[(s+1)-1]+3}{(s+1)^2+1^2} = \frac{(s+1)}{(s+1)^2+1^2} + 2\frac{1}{(s+1)^2+1^2}$$

$$= \mathcal{L}\left[e^{-t}\cos t + 2e^{-t}\sin t\right]$$

Crib sheets: Standard form: $y'' + P(x)y' + Q(x)y = \underbrace{F(x)}$

from St. Form!

Var. of Par: Given homog sol^n: $y_c = c_1 y_1 + c_2 y_2$

Get $y_p = u_1 y_1 + u_2 y_2$

$$\boxed{u_1' = \frac{-y_2 F}{W[y_1, y_2]} \qquad u_2' = \frac{y_1 F}{W[y_1, y_2]}}$$

$$W = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

integrate. No $+C$'s.

System version: $\mathbb{X} = [\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n]$ ← homog sol^{n}s.

$\vec{x}_p = \mathbb{X}\vec{u}$ where $\boxed{\mathbb{X}\vec{u}' = \vec{g}}$ $\quad \vec{u}' = \mathbb{X}^{-1}\vec{g}$

Sys: $\vec{x}' = A\vec{x} + \vec{g}$