Review problems for Exam 1 in class on Mon, Sept 16

Find the general solution of the ODE

$$1 \cdot y' + 4y = e^{-2t}.$$ linear!

Standard Form!

P(t)

Linear? Bernouli? Separable? Homogeneous? Exact? If not, int. factor u = fcn of x-alone? y-alone? Numerical!

$$u(t) = e^{\int P(t)\,dt} = e^{\int 4\,dt} = e^{4t}$$

$$\underbrace{u\left[y' + 4y\right]}_{[uy]'} = u \cdot e^{-2t} = e^{2t}$$

$u = e^{4t}$

$$e^{4t} y = \int e^{2t}\,dt = \frac{1}{2}e^{2t} + C$$

$$\boxed{y = \frac{1}{2}e^{-2t} + Ce^{-4t}}$$

Find the general solution of the ODE

$$\cancel{ty' - 2y = -2t^3 e^{2t}.}$$

not S.Form!

$$\boxed{y' - \frac{2}{t} y = -2t^2 e^{2t}} \cdot \frac{1}{t^2}$$

$$u = e^{\int -\frac{2}{t}\,dt} = e^{-2\ln|t|} = e^{\ln|t|^{-2}} = \frac{1}{t^2}$$

$$\left[\frac{1}{t^2} y\right]' = -2t^2 e^{2t} \cdot \frac{1}{t^2} = -2e^{2t}$$

$$\frac{1}{t^2} y = \int -2e^{2t}\,dt = -e^{2t} + C$$

$$y = -t^2 e^{2t} + Ct^2$$

Find the general solution of

$$\frac{dy}{dx} = \frac{xy + 2x + y + 2}{(y+2)^2} = \frac{(y+2)(x+1)}{(y+2)^2}$$

$$\frac{dy}{dx} = \frac{x+1}{y+2} \longleftarrow \text{Separable!}$$

$$\int (y+2)\,dy = \int (x+1)\,dx$$

$$\tfrac{1}{2}y^2 + 2y = \tfrac{1}{2}x^2 + x + C \longleftarrow$$

Find C from $y(x_0) = y_0$ now.

$$y^2 + 4y + (\underbrace{-x^2 - 2x - 2C}_{+K}) = 0$$

worry about $y(x_0) = y_0$ to choose $\pm$

$$y = \text{Quadratic eqn.} = \frac{(\quad) \pm \sqrt{\qquad}}{2 \cdot (\quad)}$$

Find the general solution of

$$\frac{dy}{dx} = \frac{3y-2x}{y}. = 3 - 2\frac{x}{y}$$

$$= 3 - \frac{2}{\left(\frac{y}{x}\right)} \leftarrow \text{homog!}$$

$$= 3 - \frac{2}{v} \quad \text{easy!}$$

Separable! $\boxed{x\frac{dv}{dx} + v = 3 - \frac{2}{v}}$

Let $v = \frac{y}{x}$

Solve for $y = xv$.

So $\frac{dy}{dx} = x\frac{dv}{dx} + \underbrace{\frac{dx}{dx}}_{1} \cdot v = x\frac{dv}{dx} + v$

$$x\frac{dv}{dx} = 3 - \frac{2}{v} - v = \frac{3v - 2 - v^2}{v}$$

$$\int \frac{v}{\underbrace{v^2 - 3v + 2}} dv = -\int \frac{dx}{x}$$

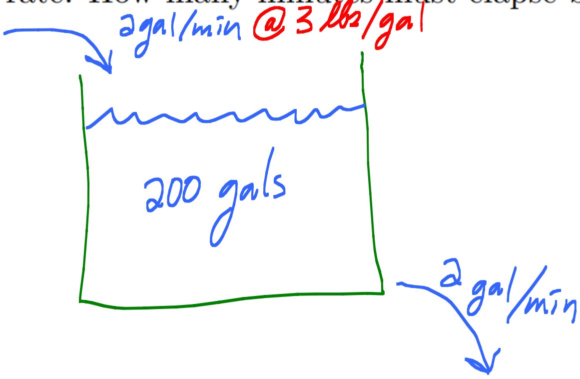
$$\frac{v}{(v-2)(v-1)} \qquad = -\ln|x| + C$$

$$\frac{A}{v-2} + \frac{B}{v-1} \qquad \text{etc.}$$

Solve for $v = $ fcn of x.

don't forget that $v = \frac{y}{x} \leftarrow$ want y!

Problem 5. A tank contains 200 *gal* (gallons) of liquid. Initially, the tank contains pure water. At time $t = 0$, brine containing 3 lb/gal salt begins to pour into the tank at a rate of 2 gal/min, and the well-stirred mixture is allowed to drain away at the same rate. How many minutes must elapse before there are 100 lb of salt in the talk?



$S(t)$ = lbs salt in tank at time t.

Pure water at $t=0$: $\underline{\underline{S(0)=0}}$

$$\frac{dS}{dt} = (\text{Rate in}) - (\text{Rate out})$$

$$= (2\ gal/min)(3\ lbs/gal) - (2\ gals/min)\left(\frac{S(t)}{200\ gals}\right)$$

Linear!
$$\boxed{\frac{dS}{dt} + \frac{1}{100} S = 6 \qquad S(0)=0}$$

$u = e^{t/100}$

$[e^{t/100} S]' = 6e^{t/100}$

$e^{t/100} S = 600 e^{t/100} + C$

$S = 600 + Ce^{-t/100}$

$C=-600$

$100 = 600(1 - e^{-t/100})$

Solve for t.

Problem 6. What is the largest open interval in which a solution to the initial value problem

$$\underbrace{(t-1)}_{\text{zeroes of this are dangerous!}}y' + \sqrt{t+2}\,y = \frac{3}{t-3}, \qquad y(2) = -5$$

Linear!

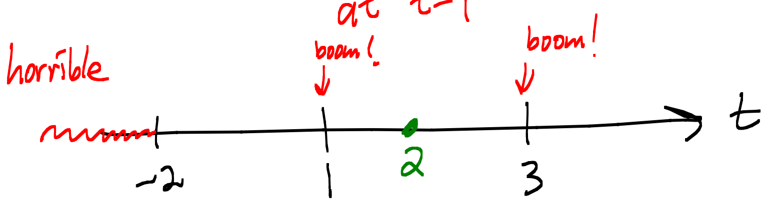
is guaranteed to exist?

horrible for $t<-2$!

Stand. Form: $$y' + \underbrace{\frac{\sqrt{t+2}}{t-1}}_{P(t)}\,y = \underbrace{\frac{3}{(t-1)(t-3)}}_{Q(t)}$$

boom! at $t=1$

boom! at $t=1,3$



E & U. Sol^n is good (continuously diff'ble and exists) on the largest interval about x_0 on which both $P(t)$ and $Q(t)$ are continuous.

Ans: $(1,3)$

Problem 7. Find all the asymptotically stable equilibrium solutions for the autonomous differential equation

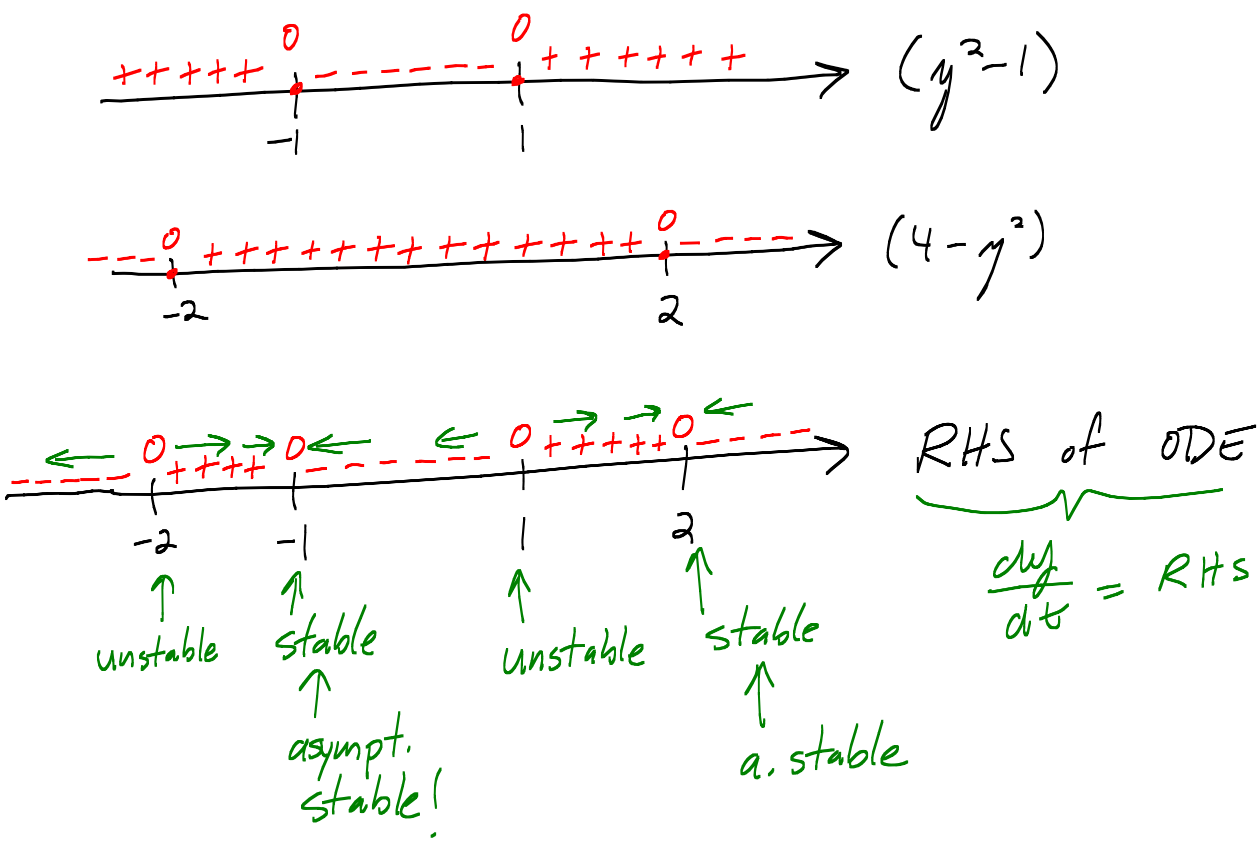
no t on RHS

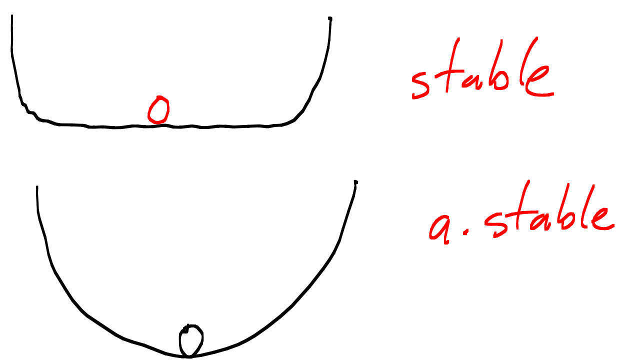
$$y' = (y^2 - 1)(4 - y^2).$$

Equil. sol^{n}s make RHS $=$ zero

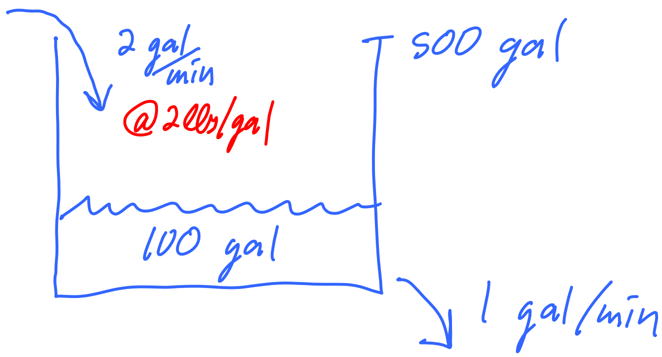
$y = \text{const}$

Equil. Sol^{n}s: $y \equiv \pm 1$ $\quad y \equiv \pm 2$





Problem 8. A tank with capacity 500 *gal* originally contains 100 gallons of water with a salt concentration of $1/2\ lb/gal$. A solution containing a salt concentration of $2\ lb/gal$ enters at a rate of $2\ gal/min$ and the well-stirred mixture is pumped out at the rate of $1\ gal/min$. What is the amount of salt in the tank after 50 *min*.



$$Q(0) = \left(\tfrac{1}{2}\right)(100) = 50$$

$$\frac{dQ}{dt} = (2 \cdot 2) - (1)\frac{Q(t)}{\underbrace{V(t)}_{100+t}}$$

Linear!

$$\frac{dV}{dt} = (\text{Rate in}) - (\text{Rate Out}), \quad V(0) = 100$$

$$= 2 - 1 = 1$$

$$V(t) = 100 + t$$

Problem 9. A skydiver weighing 200 lb (with mass 25/4 lb) falls vertically downward from an altitude of 4000 ft and opens the parachute after 10 seconds of free fall. Assume that the force of air resistance, which is directed opposite to the velocity, is 0.8 $|v|$ when the parachute is closed and $12|v|$ when the parachute is open, where the velocity v is measured in ft/sec. Use $g = 32 ft/sec^2$.

← slugs

(a) Find the speed of the skydiver when the parachute openes.

(b) Find the distance fallen before the parachute opens.

y

0 at 4000 ft

$$F = ma = m\frac{dv}{dt} \qquad v(0)=0,\ y(0)=0$$

$$-mg - 0.8\,v = m\frac{dv}{dt} \quad \leftarrow \text{Linear!} \quad \text{Get } v$$

$$\frac{dv}{dt} + \frac{.8}{m}v = -g = -32 \quad \leftarrow u = e^{\int \frac{.8}{m}dt} = e^{16t/125}$$

$$\uparrow \frac{(4/5)}{(25/4)} = \frac{16}{125}$$

$$e^{16t/125}\,v = -\int 32\, e^{16t/125}\,dt = -250\, e^{16t/125} + C \qquad \left[\begin{array}{l} v(0)=0 \text{ gives} \\ C = 250 \end{array}\right]$$

$$v = 250\left(-1 + e^{-16t/125}\right). \qquad \text{a) } v(10) = 250\left(-1 + e^{-160/125}\right)$$

$$y = \int v\,dt = 250\left(-t - \frac{125}{16}e^{-16t/125}\right) + C \qquad \left[\begin{array}{l} y(0)=0 \text{ gives} \\ C = \frac{250\cdot 125}{16} \end{array}\right]$$

$$y = 250\left(-t - \frac{125}{16}e^{-16t/125} + \frac{125}{16}\right)$$

$$\text{b) } y(10) = 250\left(-10 - \frac{125}{16}\left(e^{-160/125} - 1\right)\right)$$

Problem 10. Find the implicit solution to the initial value problem

$$\underbrace{(e^x \sin(y) - 2y\sin(x) - 1)}_{M} + \underbrace{(e^x \cos(y) + 2\cos(x) + 3)}_{N}y' = 0, \qquad y(0) = \pi.$$

$\frac{\partial M}{\partial y} = e^x \cos y - 2\sin x \qquad \frac{\partial N}{\partial x} = e^x \cos y - 2\sin x = \frac{\partial M}{\partial y}$ ✓ Exact.

(A) $\frac{\partial \varphi}{\partial x} = M$

(B) $\frac{\partial \varphi}{\partial y} = N$

(A) gives: $\varphi = \int e^x \sin y - 2y\sin x - 1 \, dx$

$= e^x \sin y + 2y\cos x - x + C(y)$

Now (B) yields: $\frac{\partial}{\partial y}\left[e^x \sin y + 2y\cos x - x + C(y)\right] = e^x \cos y + 2\cos x + 3$ ← want

$e^x \cos y + 2\cos x + C'(y)$

$C'(y) = 3$

So $\boxed{C(y) = 3y}$

Sol^n

$\varphi = e^x \sin y + 2y\cos x - x + 3y = c$

What is the general solution to this differential equation?

$$\frac{dy}{dx} = (\underbrace{x - y}_{v = x-y})^2 + 1.$$

$$\boxed{\frac{dv}{dx} = 1 - \frac{dy}{dx}}$$

$$1 - \frac{dv}{dx} = v^2 + 1$$

$$\frac{dy}{dx} = \underbrace{(x-y)^2 + 1}_{v}$$

$$1 - \frac{dv}{dx} = v^2 + 1$$

$$\frac{dv}{dx} = -v^2$$

$$\int -v^{-2}\, dv = \int dx$$

$$\frac{1}{v} = x + C$$

$$v = \frac{1}{x + C}$$

$$x - y$$

$$\boxed{y = x - \frac{1}{x + C}}$$