Review for Exam 2 (in class Wed 11/13)

2. (10 pts) Find the solution to

$$y'' + 5y' + 6y = e^{-t} = F(t)$$

$r^2 + 5r + 6 = 0$

satisfying $y(0) = 0$ and $y'(0) = 0$.

$(r+2)(r+3) = 0$

$r = -2, -3$

Homog sol^n: $c_1 e^{-2t} + c_2 e^{-3t}$

Need $y_p = Ae^{-t}$ ← If $F(t) = e^{-2t}$. Take $y_p = t(Ae^{-2t})$

$(+Ae^{-t}) + 5(-Ae^{-t}) + 6(Ae^{-t}) \overset{\text{want}}{=} e^{-t}$

$2Ae^{-t} = e^{-t}$ $\boxed{A = \frac{1}{2}}$

Gen^l Sol^n: $y = \underline{c_1} e^{-2t} + \underline{c_2} e^{-3t} + \frac{1}{2} e^{-t}$

I.C. $y' = -2c_1 e^{-2t} - 3c_2 e^{-3t} - \frac{1}{2} e^{-t}$

want

$\begin{cases} y(0) = c_1 + c_2 + \frac{1}{2} = 0 \\ y'(0) = -2c_1 - 3c_2 - \frac{1}{2} = 0 \end{cases}$

3. (10 pts) Find the general solution to

$$y'' + 2y' + y = \underbrace{\sin 2t}.$$

Homog: $y = c_1 e^{-t} + c_2 t e^{-t}$

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r = -1, -1$

$F(t) = t^2 e^{-t}$

$y_p = t^2(At^2 + Bt + C)e^{-t}$

to clear terms that solve homog.

Part. soln: $y_p = A\cos 2t + B\sin 2t$

$r = -1, -1$ $\underline{\text{Not}}$ $\pm 2i$. Safe!

$y_p' = -2A\sin 2t + 2B\cos 2t$

$y_p'' = -4A\cos 2t - 4B\sin 2t$

want

$$(-4A\cos 2t - 4B\sin 2t) + 2(-2A\sin 2t + 2B\cos 2t) + (A\cos 2t + B\sin 2t) = \sin 2t$$

$$\underbrace{[-3A + 4B]}_{=0}\cos 2t + \underbrace{[-4A - 3B]}_{=1}\sin 2t = \sin 2t$$

4. (20 pts) Given that $y = c_1 t + c_2 t^{-3}$ is the general solution to $t^2y'' + 3ty' - 3y = 0$ for $t > 0$, use the method of variation of parameters to find a particular solution to

$$\cancel{t^2}y'' + \frac{3ty'}{t^2} - \frac{3y}{t^2} = \frac{16t.}{t^2} \quad F(t) = \frac{16}{t}$$

Euler eqn. Try t^r.

$r = 1, -3$

Var. of Par. $y_p = u_1 y_1 + u_2 y_2$

where $\boxed{u_1' = \frac{-y_2 F}{W} \qquad u_2' = \frac{y_1 F}{W}}$

$$W = \det\begin{bmatrix} t & t^{-3} \\ 1 & -3t^{-4} \end{bmatrix} = -3t^{-3} - t^{-3} = -4t^{-3}$$

$$u_1' = \frac{-(t^{-3})\frac{16}{t}}{-4t^{-3}} = \frac{4}{t} \qquad u_1 = \int \frac{4}{t}\,dt = 4\ln|t| \underbrace{}_{\text{no } +C}$$

$$u_2' = \frac{(t)\frac{16}{t}}{-4t^{-3}} = -4t^3 \qquad u_2 = \int -4t^3\,dt = -t^4$$

$$y_p = (4\ln|t|)t + (-t^4)t^{-3}$$

Gen^l Sol^n: $\quad c_1 t + c_2 t^{-3} + (\underbrace{-t}_{} + 4t\ln|t|)$

dump this.
or absorb into $c_1 t$

5. (20 pts) $y = 1 = x^0$ is one solutions to

$$x^3 y''' - 12xy' = 0$$

$y = x^r$
$y' = rx^{r-1}$
$y'' = r(r-1)x^{r-2}$
$y''' = r(r-1)(r-2)x^{r-3}$

of the form x^r for $x > 0$. Find all solutions of this form. What is the general solution to this equation for $x > 0$? How do you *know* it is the general solution?

want

$$x^3\left[r(r-1)(r-2)x^{r-3}\right] - 12x\left[rx^{r-1}\right] = 0$$

$$\underbrace{\left[r(r-1)(r-2) - 12r\right]}_{\text{need } = 0} x^r = 0$$

$$r\left[(r-1)(r-2) - 12\right] = 0$$

$$r(r^2 - 3r - 10) = 0$$

$$r(r-5)(r+2) = 0 \qquad r = 0, 5, -2$$

$$y = c_1 + c_2 x^5 + c_3 x^{-2}$$

Gen^l Sol^n? Check $W = \det\begin{bmatrix} 1 & x^5 & x^{-2} \\ 0 & 5x^4 & -2x^{-3} \\ 0 & 20x^3 & 6x^{-4} \end{bmatrix}$

$$= (+1)\, 1 \cdot \det\begin{bmatrix} 5x^4 & -2x^{-3} \\ 20x^3 & 6x^{-4} \end{bmatrix} = 30 + 40 = 70$$

not zero

Gen^l Sol^n ✓

6. (20 pts) What is the correct FORM of a particular solution of

$$y^{(4)} - y''' - y'' + y' = \underbrace{t^2}_{F_1} + \underbrace{te^t}_{F_2}$$

used in the method of undetermined coefficients?
Hint: $r^4 - r^3 - r^2 + r = r(r-1)^2(r+1)$

$y_p = y_{p_1} + y_{p_2}$

for F_1 for F_2

$r = 0, +1, +1, -1$

$1 = e^0, e^t, te^t, e^{-t}$

$$y_p = t(At^2 + Bt + C) + t^2(Dt + E)e^t$$

$C = Ct^0$ solves homog

Dte^t Ee^t both solve homog

7. (10 pts) What is the general solution to

$$y^{(4)} - 16y = 0?$$

$$r^4 - 16 = 0$$

$$(r^2-4)(r^2+4) = 0$$

$$(r-2)(r+2)(r^2+4) = 0$$

$$r = \pm 2, \pm 2i \qquad e^{2it} = \underline{\cos 2t} + i\,\underline{\sin 2t}$$

$$e^{2t}, e^{-2t}, \cos 2t, \sin 2t$$

$$y = \underbrace{c_1 e^{2t} + c_2 e^{-2t}}_{c_1 \cosh 2t + c_2 \sinh 2t} + c_3 \cos 2t + c_4 \sin 2t$$

$\vec{x}' = A\vec{x} + \begin{pmatrix}3\\4\end{pmatrix}e^{-2t} \leftarrow$ If $r=-2$ is an e.val, use var of par. $\quad 2\times 2$

Fund matrix: $\mathbb{X} = [\vec{x}_1, \vec{x}_2]$

$c_1\vec{x}_1 + c_2\vec{x}_2$ is gen' sol'n to homg syst.

$\vec{x}_p = \mathbb{X}\vec{u}$ where $\mathbb{X}\vec{u}' = \begin{pmatrix}3\\4\end{pmatrix}e^{-2t} \leftarrow \vec{g}$

Get $\begin{pmatrix}u_1'\\u_2'\end{pmatrix}$. Integrate each one to get u_1, u_2. [No +C's]

Then $\vec{x}_p = u_1\vec{x}_1 + u_2\vec{x}_2$

Reduction of order: $1\,y'' + e^x y' = 0$

$P(x)$

See $y_1 = 1$ solves eqn.

Need y_2. Try $y_2 = u y_1$.

Need $u' = \frac{1}{y_1^2} e^{-\int P\,dx}$