Solutions and grading key for Exam 1, MA 366

1a. $-x\dfrac{dp}{dx} = \dfrac{V_R}{V_D}\sqrt{1+p^2} \leftarrow$ 2 pts

is separable: $\dfrac{dp}{\sqrt{1+p^2}} = -\dfrac{V_R}{V_D}\dfrac{1}{x}\,dx$

integrate: $\displaystyle\int \frac{dp}{\sqrt{1+p^2}} = -\frac{V_R}{V_D}\int \frac{1}{x}\,dx$

$$\sinh^{-1} p = -\frac{V_R}{V_D}\ln|x| + C$$

$p(a) = \dfrac{dy}{dx}(a) = 0$, so $\underbrace{\sinh^{-1} 0}_{0} = -\dfrac{V_R}{V_D}\ln|a| + C$

and we get $\boxed{C = \dfrac{V_R}{V_D}\ln|a|} \leftarrow$ 2 pts

Solve for p: $p = \sinh\left(-\dfrac{V_R}{V_D}\ln|x| + \dfrac{V_R}{V_D}\ln|a|\right)$

1b. $\dfrac{dy}{dx} = \sinh\left(\dfrac{V_R}{V_D}\underbrace{\left[\ln|a| - \ln|x|\right]}_{=\ln\left|\frac{a}{x}\right|}\right)$

$$= \sinh\left(\ln\left|\frac{a}{x}\right|^{V_R/V_D}\right)$$

$$= \frac{1}{2}\left(\exp\left(\ln\left|\frac{a}{x}\right|^{V_R/V_D}\right) - \exp\left(-\ln\left|\frac{a}{x}\right|^{V_R/V_D}\right)\right)$$

$$= \frac{1}{2}\left[\left|\frac{a}{x}\right|^{V_R/V_D} - \underbrace{\left|\frac{x}{a}\right|^{V_R/V_D}}_{\text{since } -\ln b = \ln b^{-1}}\right]$$

Taking $a=-1$ and $V_R = V_D$ gives

$$\frac{dy}{dx} = \frac{1}{2}\left[\,|x|^{-1} - |x|\,\right] = \underbrace{\frac{1}{2}\left[\frac{1}{(-x)} - (-x)\right]}_{4 \text{ pts}}$$

$$= \frac{1}{2}\left(x - \frac{1}{x}\right)$$

So $y = \frac{1}{2}\int x - \frac{1}{x}\,dx = \frac{1}{2}\left(\frac{1}{2}x^2 - \ln|x|\right) + C$

Since $y(a)=0$ when $x=a$, we find

$$0 = \frac{1}{2}\left(\frac{1}{2}a^2 - \ln|a|\right) + C$$

So $C = -\frac{1}{4}a^2 + \frac{1}{2}\ln|a| = -\frac{1}{4}(-1)^2 + \frac{1}{2}\underbrace{\ln 1}_{=0} = -\frac{1}{4}$

and we get $y = \underbrace{\frac{1}{4}x^2 - \frac{1}{2}\ln(-x)}_{2 \text{ pts.}} - \underbrace{\frac{1}{4}}_{1 \text{ pt.}}$

Since $\lim_{x \to 0^-} y(x) = \frac{1}{4}\cdot 0^2 - \frac{1}{2}(-\infty) - \frac{1}{4} = \infty$,

the dog does <u>not</u> catch the rabbit. ← 1 pt.

1c. Taking $a=-1$, $v_D = 2v_R$ gives

$$\frac{dy}{dx} = \frac{1}{2}\left[|x|^{-1/2} - |x|^{1/2}\right]$$

$$= \underbrace{\frac{1}{2}\left((-x)^{-1/2} - (-x)^{1/2}\right)}_{\text{4 pts.}}$$

So $y = \frac{1}{2}\int (-x)^{-\frac{1}{2}} - (-x)^{1/2}\,dx$

$$= \frac{1}{2}\left[-2(-x)^{1/2} + \frac{2}{3}(-x)^{3/2}\right] + C$$

Plugging in $y=0$ when $x=a$ gives

$$0 = \frac{1}{2}\left[-2(-a)^{1/2} + \frac{2}{3}(-a)^{3/2}\right] + C$$

and $C = \frac{1}{2}\left[2(-a)^{1/2} - \frac{2}{3}(-a)^{3/2}\right] = \frac{1}{2}\left[2 - \frac{2}{3}\right] = \frac{2}{3}$.

So $y = \underbrace{\frac{1}{2}\left[-2(-x)^{1/2} + \frac{2}{3}(-x)^{3/2}\right]}_{\text{2pts}} + \underbrace{\frac{2}{3}}_{\text{1 pt}}$

The dog catches the rabbit when $x=0$ and

$$y=\frac{1}{2}\left[0+0\right]+\frac{2}{3}=\frac{2}{3}, \text{ i.e., at } \underbrace{\left(0,\tfrac{2}{3}\right)}_{1\text{ pt.}}.$$

2. Say $h_1(x_0)=h_2(x_0)=y_0$.

h_1 and h_2 are solutions to $\frac{dy}{dx}=f(x,y)$, so

$$h_1'(x_0)=f(x_0,h_1(x_0))=f(x_0,y_0)$$

and

$$h_2'(x_0)=f(x_0,h_2(x_0))=f(x_0,y_0)$$

same thing

Hence, it cannot happen that $h_1'(x_0)\neq h_2'(x_0)$, i.e., graphs of solutions cannot cross. 10 pts.

For the second part of the question, the Existence and Uniqueness Theorem 2.4.2 on p. 52 says that if $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are continuous on the xy-plane, then if h_1 and h_2 both solve the initial value problem $\frac{dy}{dx}=f(x,y)$ with $y(x_0)=y_0$, i.e., if $h_1(x_0) = h_2(x_0) = y_0$, then h_1 and h_2 must be the same (unique) solution. 10 pts

[You could also give any condition that implies that $\frac{\partial f}{\partial y}$ is continuous, for example f is C^1-smooth (continuously differentiable) or C^2-smooth.]

3. Let $u = xy$. Then $y = x^{-1} u$ and

$$\frac{dy}{dx} = \underbrace{-x^{-2} u + x^{-1} \frac{du}{dx}}_{10 \text{ pts}} \quad \text{via the product rule}$$

Plugging into the ODE:

$$\frac{dy}{dx} = \sin(xy)$$

$$-x^{-2} u + x^{-1} \frac{du}{dx} = \sin(u)$$

and solving for $\frac{du}{dx}$ yields

$$\frac{du}{dx} = x\left(\sin u + x^{-2}u\right)$$

$$\frac{du}{dx} = \underbrace{\frac{u}{x} + x\sin u}_{F(x,u)} \leftarrow \text{10 pts}$$

4a. Standard form: $y' + \frac{1}{2}y = \frac{1}{2}e^{-5x}$

Int. factor: $u = e^{\int \frac{1}{2}dx} = e^{x/2}$

Use if: $\underbrace{e^{x/2}\left(y' + \frac{1}{2}y\right)}_{\left[e^{x/2}y\right]'} = e^{x/2}\cdot\frac{1}{2}e^{-5x}$

$$\left[e^{x/2}y\right]' = \frac{1}{2}e^{-9x/2}$$

$$e^{x/2}y = \frac{1}{2}\int e^{-9x/2}dx = -\frac{1}{9}e^{-9x/2} + C$$

$$y = -\frac{1}{9}e^{-5x} + Ce^{-x/2}$$

Plugging in $x=0$ and $y=1$ yields $1 = -\frac{1}{9} + C$ $\boxed{C = 10/9}$

Soln: $\underline{y = -\frac{1}{9}e^{-5x} + \frac{10}{9}e^{-x/2}}$ $\leftarrow$ 10 pts

b) $y' + \frac{1}{2}y = \frac{1}{2}g(x)$

$$e^{x/2}\left(y' + \frac{1}{2}y\right) = \frac{1}{2}e^{x/2}g(x)$$

$$\left[e^{x/2}y\right]' = \frac{1}{2}e^{x/2}g(x)$$

$$e^{x/2}y = \underbrace{\frac{1}{2}\int_0^x e^{t/2}g(t)\,dt}_{\substack{\text{antiderivative of } e^{x/2}g(x) \\ \text{by Fund. Thm. Calc.}}} + C$$

Plug in $x=0$ and $y=0$ to find $C=0$. $\left(\int_0^0 = 0 \text{ area}\right)$

$$\underbrace{y = \frac{1}{2}e^{-x/2}\int_0^x e^{t/2}g(t)\,dt}_{10\text{ pts}} = \int_0^x \underbrace{\frac{1}{2}e^{(t-x)/2}}_{K(x,t)}g(t)\,dt$$

and we see that $y(x) =$ (area under a curve) which depends on all the values of g between 0 and x.

4. Multiplying by x^{-3} gives the new ODE

$$\underbrace{(2x^{-2} - x^{-3}y^2)}_{M} + \underbrace{(x^{-2}y)}_{N}\frac{dy}{dx} = 0$$

Test for exactness:

4pts
$$\begin{cases} \dfrac{\partial}{\partial y}\left[2x^{-2} - x^{-3}y^2\right] \overset{?}{=} \dfrac{\partial}{\partial x}\left[x^{-2}y\right] \\[2ex] 0 - 2x^{-3}y \overset{?}{=} -2x^{-3}y \quad \text{Yes.} \end{cases}$$

We need $\varphi(x,y)$ with

$$\frac{\partial \varphi}{\partial x} = M = 2x^{-2} - x^{-3}y^2 \qquad (A)$$

$$\frac{\partial \varphi}{\partial y} = N = x^{-2}y \qquad (B)$$

Using (A): $\varphi = \int 2x^{-2} - x^{-3}y^2 \, dx + C(y)$

$$\varphi = \underbrace{-2x^{-1} + \frac{1}{2}x^{-2}y^2}_{2\text{ pts}} + \underbrace{C(y)}_{\substack{\text{arb. fcn. of } y \\ 3\text{pts}}}$$

Next, use (B) to pin down $C(y)$:

$$\frac{\partial}{\partial y}\left[-2x^{-1} + \frac{1}{2}x^{-2}y^2 + C(y)\right] = \underbrace{x^{-2}y}_{N}$$

want

$$0 + x^{-2}y + C'(y) = x^{-2}y$$

$$\boxed{c'(y) = 0}$$ So $c(y)$ is a constant.

We take $c(y) \equiv 0$ (or c_1 if you prefer)

We now have $\underbrace{\varphi(x,y) = -2x^{-1} + \frac{1}{2}x^{-2}y^2 + 0}_{5\text{ pts}}$

and the general solution is $\varphi(x,y) = k$

$$\boxed{-2x^{-1} + \frac{1}{2}x^{-2}y^2 = K}$$ 3 pts.

← defines $y = \text{fcn of } x$ implicitly

((Can solve for y here if you like:

$$y = \pm\sqrt{2x^2\left(K + \frac{2}{x}\right)}\;)$$

Finally, plugging $x=2$ and $y=4$ in the box:

$$-2\cdot 2^{-1} + \frac{1}{2}2^{-2}\cdot 4^2 = K$$

$$K = -1 + 2 = 1 \qquad \boxed{K=1}$$

and we get

$$-2x^{-1} + \frac{1}{2}x^{-2}y^{2} = 1$$

$$y^2 = 2x^2\left(1 + \frac{2}{x}\right)$$

$$y = \pm\sqrt{2x^2 + 4x}$$

and we must take the positive square root to have $y(2) = 4$:

$$\boxed{y = +\sqrt{2x^2 + 4x}}$$

3pts