MATH 366

Exam 1

1. (20 pts) Find the solution to $\frac{1}{2}\frac{dy}{dx} + y = 3e^{4x}$ satisfying $y(0) = 3$.

2. (20 pts) Find the general solution to the exact equation

$$(3x^2y^2 + 1) + (2x^3y + 2y)\frac{dy}{dx} = 0.$$

Write your answer in the form $y =$(*a function of* x) if you can.

3. (20 pts) Solve the initial value problem

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}$$

with $y(1) = -2$ by making the substitution $v = y/x$. Write your answer in the form $y =$(*a function of* x) if you can.

4. (20 pts) A large leaky fish farm tank initially containing 1,000 gallons of pure water is filled by a hose that pumps in a solution containing 7 pounds of salt per gal at a rate of 3 gal/min. The well stirred mixture leaks out a hole in the bottom at 2 gal/min. Let $Q(t)$ denote the number of pounds of salt in the tank at time t (in minutes). **SET UP BUT DO NOT SOLVE** an initial value problem for $Q(t)$, i.e., find an ODE plus an initial condition that determines $Q(t)$.

5. (20 pts) For the autonomous first order equation

$$y' = y(y-1)^2(y-2),$$

a) what are the equilibrium solutions (i.e., the constant solutions)?
b) Specify the stability of each solution you found in part (a): stable, unstable, or semi-stable.
c) Which, if any, of the equilibrium solutions you found are asymptotically stable? Explain.

1. Standard Form:

$$y' + 2y = 6e^{4x}$$

$$u = e^{\int 2\,dx} = e^{2x}$$

$$\underbrace{e^{2x}(y' + 2y)} = 6e^{4x}e^{2x} = 6e^{6x}$$

$$[e^{2x}y]'$$

So $\quad e^{2x}y = \int 6e^{6x}\,dx = e^{6x} + C$

and $\quad y = e^{4x} + Ce^{-2x}$

$$y(0) = e^{0} + Ce^{0} \overset{\text{want}}{=} 3 \quad \boxed{C=2}$$

Ans. $\boxed{y = e^{4x} + 2e^{-2x}}$

2. $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(2x^3y + 2y\right) = 6x^2y$

$\stackrel{\checkmark}{=}$

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(3x^2y^2+1\right) = 6x^2y$

Need $\frac{\partial \varphi}{\partial x} = M = 3x^2y^2+1 \quad (A)$

$\frac{\partial \varphi}{\partial y} = N = 2x^3y + 2y \quad (B)$

Use (A): $\varphi = \int 3x^2y^2+1\,dx = x^3y^2 + x + C(y)$

Use (B): $\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y}\left[x^3y^2 + x + C(y)\right] \stackrel{\text{want}}{=} 2x^3y + 2y$

$2x^3y + C'(y) = 2x^3y + 2y$

$C'(y) = 2y$

$C(y) = y^2$

$\varphi = x^3y^2 + x + y^2 = y^2(x^3+1) + x = K$

$y = \pm\sqrt{\frac{K-x}{x^3+1}}$

3. $$\frac{dy}{dx} = \frac{1}{\left(\frac{y}{x}\right)} + \left(\frac{y}{x}\right)$$

$$v + x\frac{dv}{dx} = \frac{1}{v} + v$$

$$v\,dv = \frac{dx}{x}$$

$$\int v\,dv = \int \frac{dx}{x}$$

$$\frac{1}{2}v^2 = \ln|x| + C$$

$$\frac{1}{2}\left(\frac{y}{x}\right)^2 = \ln|x| + C \leftarrow$$ Want $y(1) = -2$

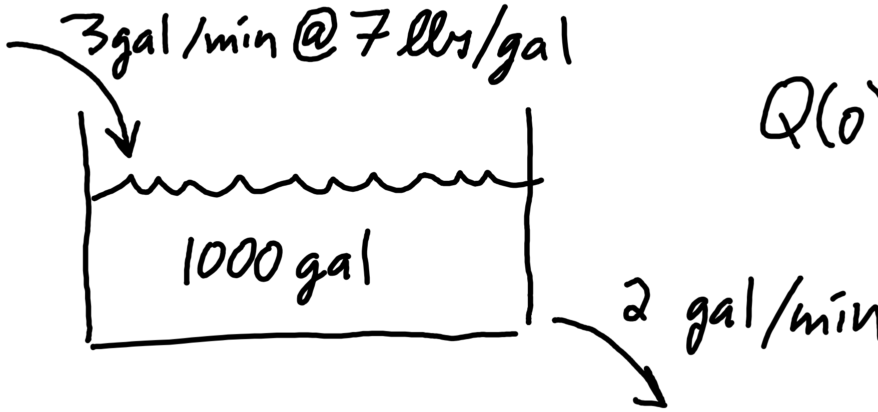
$$\frac{1}{2}\left(\frac{-2}{1}\right)^2 = \underbrace{\ln 1}_{0} + C \qquad \boxed{C=2}$$

$$\frac{y}{x} = \pm\sqrt{2\ln|x| + 2\cdot 2}$$

Need $-$ to make $y(1) = -2$.

$$\boxed{y = -x\sqrt{4 + \ln x^2}}$$

4.



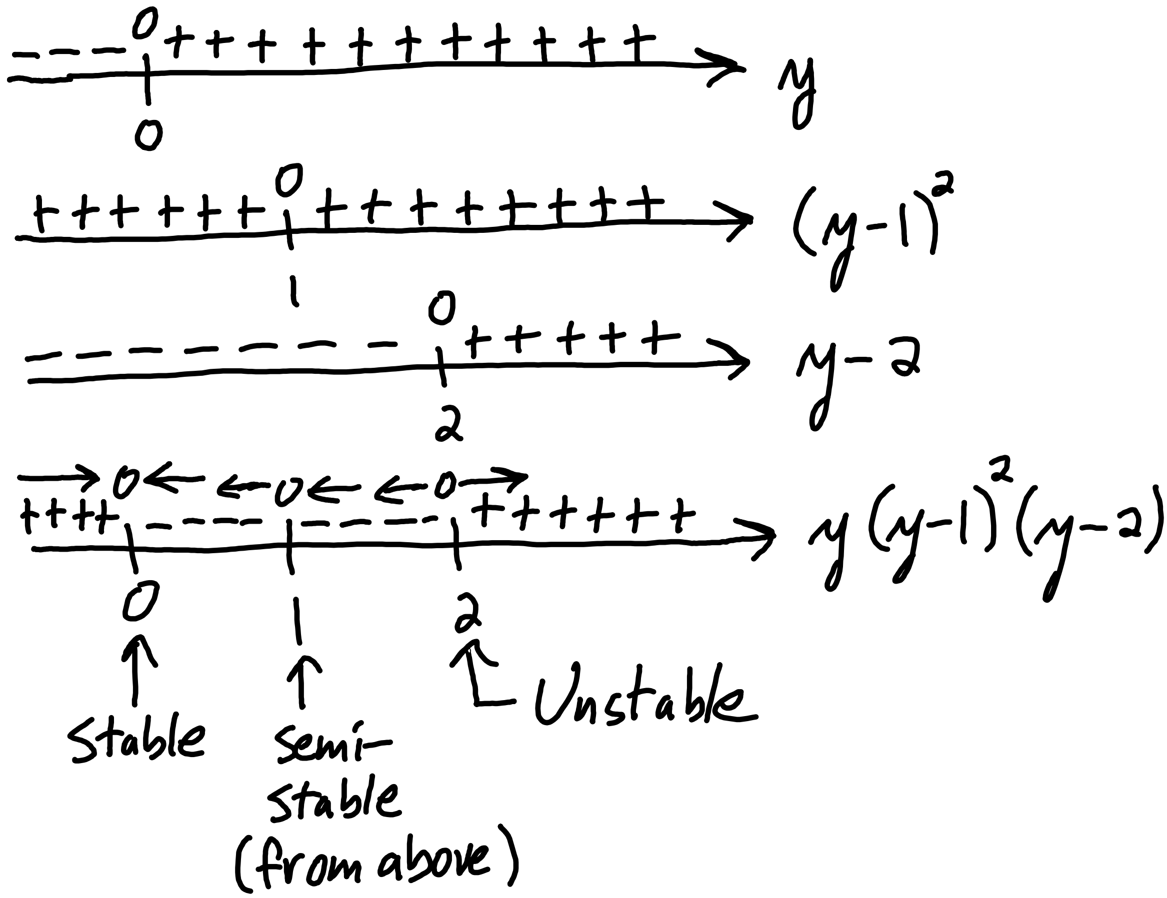
$Q(0)=0$ ("pure")

$$\frac{dQ}{dt} = (3\,\text{gal/min})(7\,\text{lbs/gal}) - (2\,\text{gal/min})\left[\frac{Q(t)}{1000+t}\right]$$

$$\boxed{\frac{dQ}{dt} = 21 - \frac{2}{1000+t}Q \qquad Q(0)=0}$$

5. a) Constant solutions: $y \equiv 0$, $y \equiv 1$, $y \equiv 2$.

b)



$y \equiv 0$ is stable

$y \equiv 1$ is semi-stable

$y \equiv 2$ is unstable

c) $y \equiv 0$ is asymptotically stable because small perturbations get pushed back toward 0.