Lesson 4 Cauchy-Riemann equations, analytic functions

Lessons 1, 2, 3 due Wed. Lessons 4, 5, 6 due the next Wed. Read Chap. 2

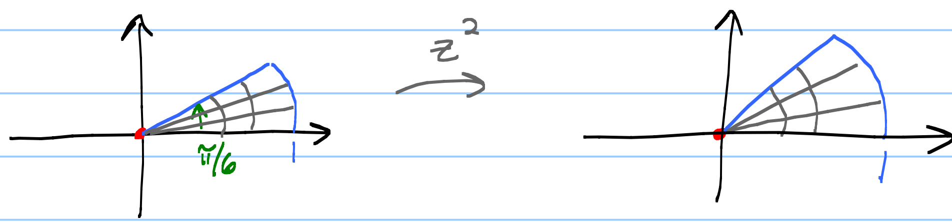
EX: $\underbrace{f(z) = \bar{z}}_{\mathbb{C} \to \mathbb{C}}$ $\qquad f(x+iy) = \overline{x+iy} = \underset{u(x,y)}{x} - i\, \underset{-y = v(x,y)}{y}$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} \qquad \mathbb{R}^2 \to \mathbb{R}^2 \quad \underline{\text{equiv}}$$

$DQ \to e^{-2i\theta}$ $\qquad z_0 \quad z_0 + re^{i\theta}$

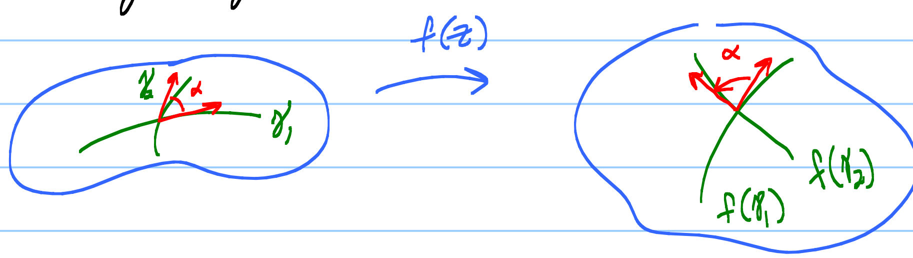
EX: $f(z) = z^2 = (x+iy)^2 = \underbrace{[x^2 - y^2]}_{u(x,y)} + i\, \underbrace{[2xy]}_{v(x,y)}$

$f(re^{i\theta}) = (re^{i\theta})^2 = r^2 e^{i2\theta}$ by DeMoivre's.

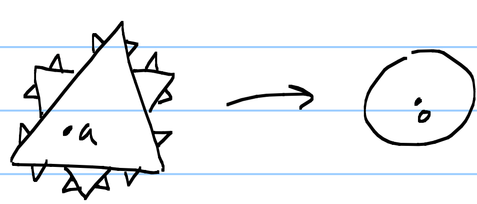


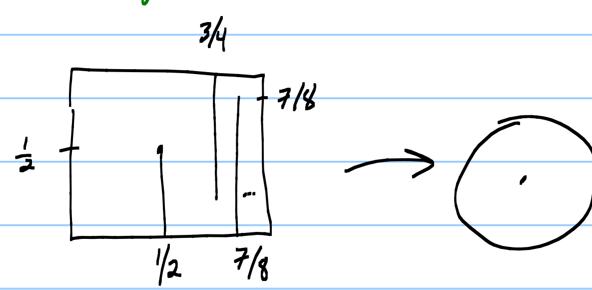
Think: Sheet of rubber. Stretched like a fan opening.

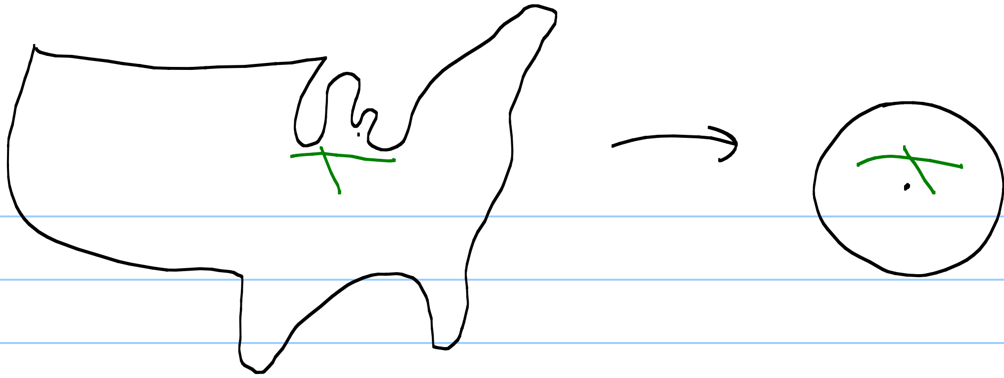
Fact: z^2 is conformal inside the fan.
Meaning angles are preserved.



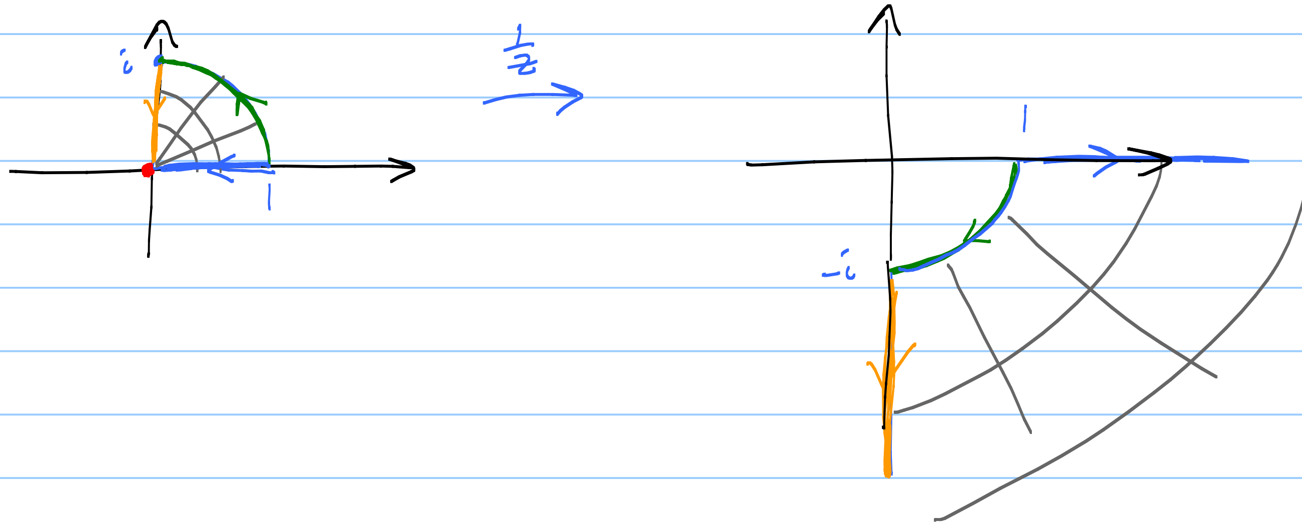
Riemann Mapping Theorem: Given a domain $\Omega \neq \mathbb{C}$ that has no "holes" (simply connected), there is a one-to-one $\underbrace{\text{complex differentiable}}_{\text{"analytic"}}$ mapping f of Ω onto the unit disc $D_1(0)$.





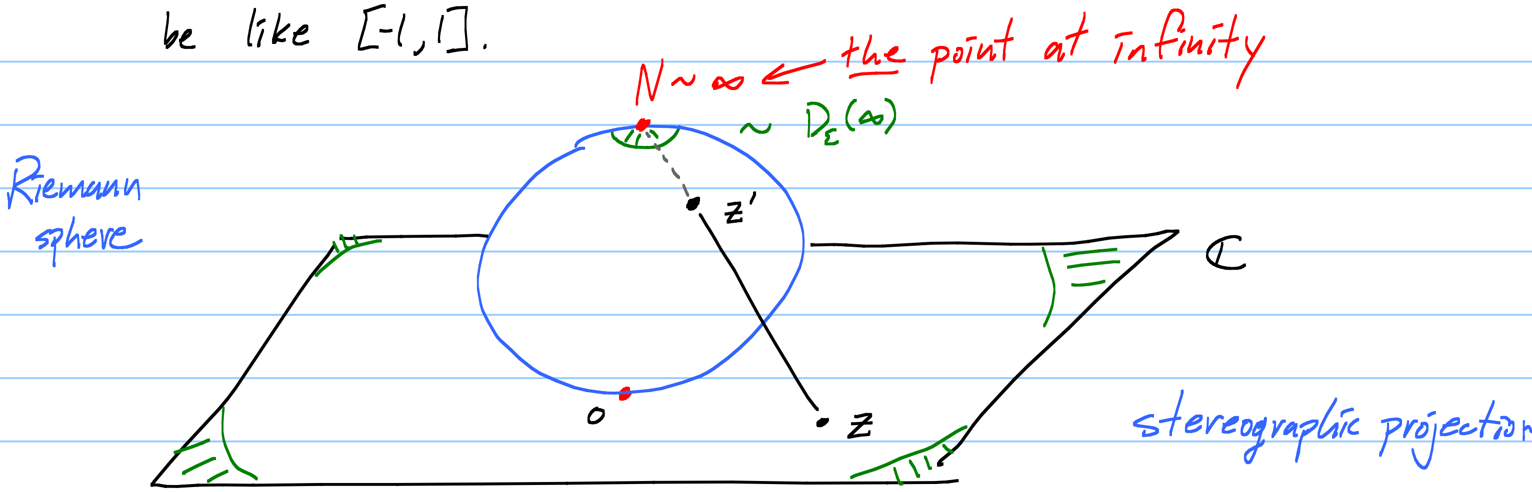


EX: $f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \left(\frac{1}{r}\right) e^{-i\theta}$



$\mathbb{R}$: $-\infty \ldots$ ——— 0 ———→ $\ldots \infty$

$[-\infty, \infty]$ is compactification of the real line. Makes it be like $[-1, 1]$.



The "point at infinity" corresponds to the North Pole N.

"∞" is an abstract point added to $\mathbb{C}$.

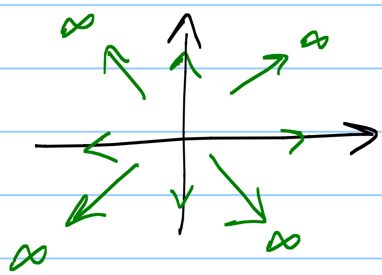
$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the "extended complex plane."

$$D_\varepsilon(\infty) = \{ z : |z| > \tfrac{1}{\varepsilon} \} \cup \{\infty\}$$

This is the way to "feel" about the point at ∞.

<u>Defn</u>: $\lim_{z \to z_0} f(z) = \infty$ means, given $M > 0$, no matter how big, there is a $\delta > 0$ such that $|f(z)| > M$ when $|z - z_0| < \delta$, $z \neq z_0$.

$M \sim \frac{1}{\varepsilon}$



<u>Def^n</u>: A function $f : \Omega \to \mathbb{C}$ on an <u>open</u> set Ω is called <u>analytic</u> if it is complex diff'ble at each point in Ω.

MA 425/525: Analytic fcns on domains.

<u>Thm</u>: Analytic fcns are continuous fcns.

<u>Why</u>: $(\dagger)\ DQ = \dfrac{f(z) - f(a)}{z - a} = f'(a) + E(z)$ ← "error" at $z \neq a$.

$f'(a)$ exists means that $\lim_{z \to a} E(z) = 0$.

Solve $(\dagger)$ for $f(z)$: $f(z) = f(a) + f'(a)(z-a) + E(z)(z-a)$

as $z \to a$: $f(a) + f'(a) \cdot 0 + 0 \cdot 0 = f(a)$ ✓

<u>Remark</u>: Mind blowing fact : f analytic $\Rightarrow$ f' analytic!