Lesson 5 Cauchy-Riemann Equations

4, 5, 6 due Wed.

Last time: Analytic functions are continuous.

Fact: If f is complex diff'ble at a, then f is continuous at a.

Proof that product rule holds:

$$DQ = \frac{f(z)g(z) - f(a)g(a)}{z-a} = \frac{[(f(z)-f(a))+f(a)]\,g(z) - f(a)g(a)}{z-a}$$

$$= \frac{f(z)-f(a)}{z-a}\cdot g(z) + f(a)\cdot\frac{g(z)-g(a)}{z-a}$$

$z \to a$ ← g is cont. at a

$$\longrightarrow f'(a)\cdot g(a) + f(a)\cdot g'(a)$$

Cauchy-Riemann Eqns: $DQ = \dfrac{f(z)-f(a)}{z-a} \longrightarrow L$

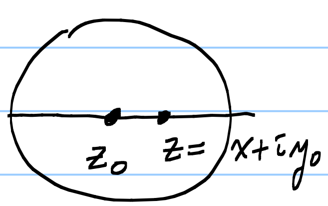
DQ for z: $e^{-i2\theta}$ different for different rays!

Idea: Check for just two directions: horiz, vert.

Notation: $f(\underbrace{x+iy}_{z}) = u(x,y) + i\,v(x,y)$ $\qquad u = u(x,y)$

$z_0 = x_0 + iy_0$ $\qquad u_0 = u(x_0,y_0)$

Step 1: Horiz

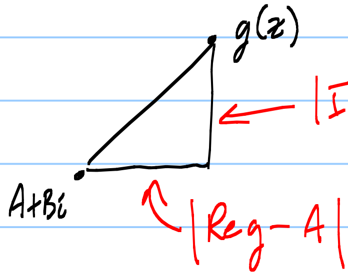


$$DQ = \frac{f(z)-f(z_0)}{z-z_0} = \frac{[u(x,y_0)+iv(x,y_0)] - [u_0+iv_0]}{(x+iy_0)-(x_0+iy_0)}$$

$$= \frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} + i\,\frac{v(x,y_0)-v(x_0,y_0)}{x-x_0} \longrightarrow f'(a)$$

Fact: $\lim\limits_{z\to z_0} g(z) = A + iB \iff \begin{cases} \lim\limits_{(x,y)\to(x_0,y_0)} \operatorname{Re} g = A \\ \lim\limits_{(x,y)\to(x_0,y_0)} \operatorname{Im} g = B \end{cases}$

Why:



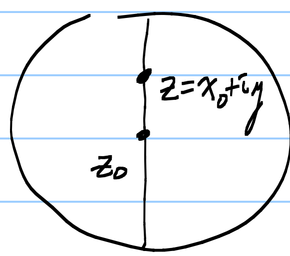
($\Rightarrow$) $|\text{Leg}| \le |\text{hypot.}|$

($\Leftarrow$) Pythagoras

Conclude: If $f'(z_0)$ exists, then $\frac{\partial u}{\partial x}(x_0,y_0)$ and $\frac{\partial v}{\partial x}(x_0,y_0)$ exist and

$$\boxed{f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\,\frac{\partial v}{\partial x}(x_0,y_0)}$$

Step 2: Vertical



$$DQ = \frac{[u(x_0,y)-u_0] + i\,[v(x_0,y) - v_0]}{(x_0+iy) - (x_0+iy_0)}$$

$$= \underbrace{\left(\frac{1}{i}\right)}_{-i} \cdot \frac{u(x_0,y)-u(x_0,y_0)}{y-y_0} + \frac{v(x_0,y)-v(x_0,y_0)}{y-y_0}$$

$$\longrightarrow \frac{\partial v}{\partial y}(x_0,y_0) - i\,\frac{\partial u}{\partial y}(x_0,y_0) \qquad \text{if } f'(z_0) \text{ exists.}$$

Conclude: If $f'(z_0)$ exists, then first partials in y of u and v exist at (x_0,y_0) and

$$\boxed{f'(z_0) = \frac{\partial v}{\partial y}(x_0,y_0) - i\,\frac{\partial u}{\partial y}(x_0,y_0)}$$

Cauchy-Riemann Equations (C-R Eqns): Equate red boxes

If f is complex diff'ble at z_0, then all the first partials of u and v exist at (x_0, y_0) and

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}} \quad \text{at } (x_0, y_0)$$

Fact: C-R Eqns hold on the open set where f is analytic.

C-R Eqns are a necessary condition for analyticity.

EX: $f(x+iy) = \overline{x+iy} = x - iy$

$u = x$ $\quad$ $v = -y$

ouch!

$$\begin{cases} \dfrac{\partial u}{\partial x} = 1 \neq -1 = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = 0 = -\dfrac{\partial v}{\partial x} \end{cases}$$

Not analytic!

EX: $E(x+iy) = \underbrace{e^x \cos y}_{u} + i\,\underbrace{e^x \sin y}_{v}$

$$\begin{cases} \dfrac{\partial u}{\partial x} = e^x \cos y \overset{\checkmark}{=} e^x \cos y = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = e^x(-\sin y) \overset{\checkmark}{=} -e^x \sin y = -\dfrac{\partial v}{\partial x} \end{cases}$$

Hmmm: C-R Eqns hold. Does that mean $E(z)$ is analytic?

Theorem: $f(x+iy) = u(x,y) + i\,v(x,y)$ on a domain Ω and

1) u and v are continuous on Ω, and

2) all the first partial derivatives of u and v exist and are continuous on Ω, and

3) u and v satisfy the C-R Eqns on Ω,

then f is <u>analytic</u> on Ω.

Note: 1 & 2 mean that u and v are C^1-smooth.

<u>Consequence</u>: $E(z)$ is analytic on $\Omega = \mathbb{C}$.

($e^x \cos y$ and $e^x \sin y$ are C^∞-smooth.)

<u>Red boxes</u>: $f'(z) = u_x + i v_x$
$= v_y - i u_y$ $\leftarrow$ $E'(z) = \underbrace{e^x \cos y + i e^x \sin y}_{E(z)\,!}$

<u>Fact</u>: $E(z)$ is analytic on $\mathbb{C}$ and $E'(z) = E(z)$.

[Adding that $E(0) = 1$ uniquely determines the complex exponential fcn $E(z)$.]

<u>Important calculus lemma</u>: If u is C^1-smooth, then

$$u(x,y) = u(x_0,y_0) + u_x(x_0,y_0)(x-x_0) + u_y(x_0,y_0)(y-y_0) + R(x,y)$$

where $$\lim_{(x,y)\to(x_0,y_0)} \frac{R(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0.$$

Assignment: Look this up.