Lesson 6

4, 5, 6 due Wed after Labor Day

Calculus lemma: u C^1-smooth

$$u(x,y) = \underbrace{u(x_0,y_0)}_{u^0} + \underbrace{u_x(x_0,y_0)}_{u_x^0}(x-x_0) + \underbrace{u_y(x_0,y_0)}_{u_y^0}(y-y_0) + R(x,y)$$

$$\frac{R(x,y)}{\underbrace{\sqrt{(x-x_0)^2+(y-y_0)^2}}_{|z-z_0|}} \longrightarrow 0 \quad \text{as} \quad (x,y) \to (x_0,y_0)$$

$z = x+iy \qquad z_0 = x_0 + iy_0$

$z \longrightarrow z_0$

Why:

$$R(x,y) = \left\{ u(x,y) - u(x_0,y_0) - u_x^0(x-x_0) - u_y^0(y-y_0) \right.$$

$$u(x,y) - u(x_0,y_0) = \int_0^1 \frac{d}{dt}\left[u\big(\underbrace{x_0+t(x-x_0),\ y_0+t(y-y_0)}_{\text{line from } (x_0,y_0) \text{ to } (x,y)}\big)\right] dt$$

$$= \int_0^1 u_x(x-x_0) + u_y(y-y_0)\, dt - \int_0^1 \underbrace{u_x^0(x-x_0) + u_y^0(y-y_0)}_{\text{const.}}\, dt$$

$$= \int_0^1 \Big[\underbrace{u_x(x_0+t(x-x_0), y_0+t(y-y_0)) - u_x^0}_{|\ \ | < \varepsilon/2}\Big](x-x_0)\, dt$$

$$+ \int_0^1 \Big[\underbrace{u_y(x_0+t(x-x_0), y_0+t(y-y_0)) - u_y^0}_{|\ \ | < \frac{\varepsilon}{2}}\Big](y-y_0)\, dt$$

Aha! Given $\varepsilon > 0$, $\exists\ \delta > 0$ such that $|z-z_0| < \delta \implies$

$$\begin{cases} |u_x(z) - u_x(z_0)| < \frac{\varepsilon}{2} \\ |u_y(z) - u_y(z_0)| < \frac{\varepsilon}{2} \end{cases}$$

$$|R(x,y)| \le \int_0^1 \frac{\varepsilon}{2}|x-x_0|\, dt + \int_0^1 \frac{\varepsilon}{2}|y-y_0|\, dt$$

Done! $$\frac{|R(x,y)|}{|z-z_0|} \le \frac{\varepsilon}{2}\underbrace{\frac{|x-x_0|}{|z-z_0|}}_{\le 1} + \frac{\varepsilon}{2}\underbrace{\frac{|y-y_0|}{|z-z_0|}}_{\le 1} < \varepsilon \quad \checkmark$$

Theorem: If u, v are C^1-smooth and satisfy the CR Eqns

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2mm] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases},$$

then $f(x+iy) = u(x,y) + i\,v(x,y)$ is analytic.

Why:

$$u = u_0 + u_x^0 (x-x_0) + u_y^0 (y-y_0) + R_u$$
$$v = \cdots\cdots + R_v$$

$$DQ = \frac{f(z)-f(z_0)}{z-z_0} = \frac{(u+iv)-(u^0+iv^0)}{z-z_0}$$

C-R Eqns

$$= \frac{u_x^0(x-x_0) + \underbrace{u_y^0}_{=-v_x^0}(y-y_0) + i\left[v_x^0(x-x_0) + \underbrace{v_y^0}_{=u_x^0}(y-y_0)\right]}{z-z_0} + \frac{R_u + iR_v}{z-z_0}$$

Aha! complex mult.

$$= \frac{\left[u_x^0 + i v_x^0\right]\left[(x-x_0) + i(y-y_0)\right]}{z-z_0} + \frac{R_u + iR_v}{|z-z_0|}$$

$$DQ = \left[u_x^0 + i\,v_x^0\right] + \mathcal{E}$$

where $$|\mathcal{E}| \le \frac{|R_u|}{|z-z_0|} + \frac{|R_v|}{|z-z_0|} \longrightarrow 0 \text{ as } z \to z_0,$$

by Taylor's Thm with remainder.

So $f'(z_0)$ exists and $f'(z_0) = u_x(x_0,y_0) + i\,v_x(x_0,y_0)$.

Fact: $E(z)$ is an <u>entire</u> function, meaning it is analytic on $\mathbb{C}$.

Theorem: Suppose u and v are $\mathcal{C}^2$-smooth and satisfy the C-R Eqns. [Then we know $u+iv$ is analytic.]

It follows from C-R eqns that u and v are <u>harmonic</u>, meaning that $\Delta u = 0$.

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

Why: (A) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \leftarrow \dfrac{\partial}{\partial x}$ (A) : $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial x \partial y}$

(B) $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \leftarrow \dfrac{\partial}{\partial y}$ (B) ; $\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial^2 v}{\partial y \partial x}$

$\mathcal{C}^2$-smooth $\Rightarrow$ mixed partials equal

Aha! $\dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial^2 u}{\partial y^2}$. $\quad \Delta u \equiv 0.$

Similarly for v.

Def^n: Given a harmonic function u, then if we can find another harmonic function v such that $u+iv$ is analytic, then v is called a <u>harmonic conjugate</u> for u.

Problem: How? Given u $\mathcal{C}^2$-smooth with $\Delta u \equiv 0$.

All we need is a v with $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

Hmmm : $\begin{cases} v_x = -u_y \\ v_y = u_x \end{cases}$ $\qquad \nabla v = \underbrace{-u_y \hat{\imath} + u_x \hat{\jmath}}_{\text{vector field } \vec{F}}$

v potential

Can find v (on a domain without holes) provided $\text{Curl}\,\vec{F} = 0$.

But $\text{Curl}\,\vec{F} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -u_y & u_x & 0 \end{bmatrix} = (\Delta u)\, \hat{k}$

We will see that we can always find a harmonic conj on a domain without holes.

<u>Easy Lemma</u>: If $f'(z) \equiv 0$ on a domain, then f is constant on the domain.

<u>Why</u>: $f' = \begin{cases} u_x + i v_x \\ -v_y + i u_y \end{cases}$ $\leftarrow$ Red boxes

$f' \equiv 0$ implies $\nabla u \equiv 0$ and $\nabla v \equiv 0$.

$$u(x,y) - u(x_0,y_0) = \int_{\gamma_{(x_0,y_0)}^{(x,y)}} \nabla u \cdot d\vec{s} = 0$$