Lecture 7 Harmonic conjugates

Lessons 7, 8 due next Wed.

Know: $f = u + iv$ analytic $\Rightarrow$ f cont $\Rightarrow$ u, v continuous.

$\Rightarrow$ u_x, u_y, v_x, v_y exist and satisfy C-R eqns

Big fact: f analytic $\Rightarrow$ f' analytic $\Rightarrow$ f'' analytic $\Rightarrow$ ...

and so u, v are C^∞-smooth and harmonic.

See Ahlfor's, Complex analysis for a proof or my MA 530 videos.

Lemma: If f analytic on a domain and $f' \equiv 0$, then f is constant on the domain.

Why:

$$f' = \begin{cases} u_x + i v_x \\ v_y - i u_y \end{cases} \equiv 0 \quad \Rightarrow \quad \begin{matrix} \nabla u \equiv 0 \\ \nabla v \equiv 0 \end{matrix}$$

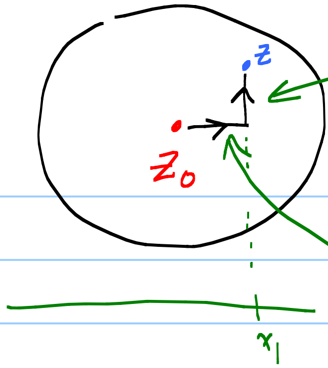
$$u(x,y) - u(x_0, y_0) = \int_{\gamma_{z_0}^{z}} \underbrace{\nabla u}_{\equiv 0} \cdot d\vec{s} = 0$$

So $u(x,y) \equiv u(x_0, y_0)$. (Same for v.)

Little problem: If we don't assume the Big Fact, we don't know u_x, u_y continuous and $\int \cdots d\vec{s}$ makes sense.

How to fix: Freshman calculus: If $h(t)$ is continuous on $[a,b]$ and diff'ble on (a,b) and $h'(t) \equiv 0$ on (a,b), then h is constant on $[a,b]$. Pf: MVT

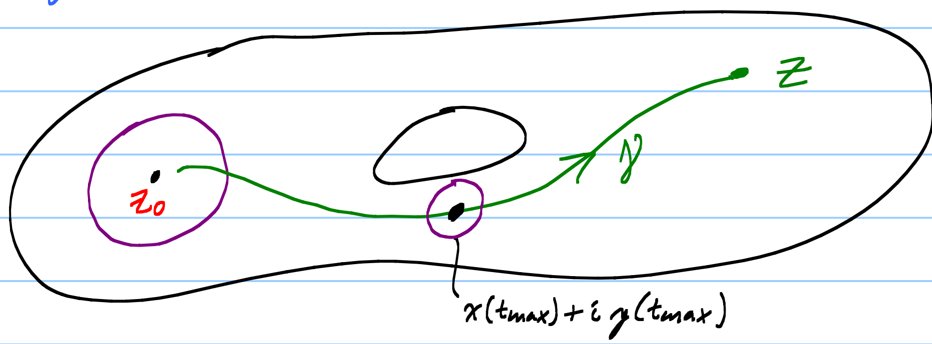
Lemma: If $f' \equiv 0$ on $D_r(z_0)$, then $f \equiv$ const on $D_r(z_0)$.



$\frac{\partial u}{\partial y}(x_1, y) \equiv 0$, $\frac{\partial v}{\partial y}(x_1, y) \equiv 0$ $\Rightarrow$ u, v const along vert.

$\frac{\partial u}{\partial x}(x, y_0) \equiv 0$, $\frac{\partial v}{\partial x}(x, y_0) \equiv 0$ Fresh. Calc $\Rightarrow$ u, v const along horiz.

How to go from disc to domain:



γ: $(x(t), y(t))$
$a \le t \le b$.

See that f const c on a disc at z_0.

Let t_{max} = supremum of t such that $f(x(s)+i\,y(s)) = c$ for all s with $a \le s \le t$.

Claim: $t_{max} = b$. Why: If $t_{max} < b$, stick in a disc at $x(t_{max}) + i\,y(t_{max})$ and see t_{max} is not t_{max}!

Assume Big Fact from now on. So f analytic $\Rightarrow$ u, v harmonic.

Problem: Given harmonic C^2-smooth function u, how do we cook up a harmonic conjugate function v such that $u + iv$ is analytic.

Prob: Given u with $\Delta u \equiv 0$ find v such that

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad \text{i.e.,} \quad \left.\begin{aligned} v_x &= -u_y \\ v_y &= u_x \end{aligned}\right\} \quad \vec{F} = -u_y\,\hat{\imath} + u_x\,\hat{\jmath}$$

$\underbrace{}_{\nabla v}$

v is a potential for $\vec{F}$.

$$\text{Curl } \vec{F} = \left[\frac{\partial}{\partial x}(y \text{ comp. of } \vec{F}) - \frac{\partial}{\partial y}(x \text{ comp of } \vec{F})\right]\hat{k} = (\Delta u)\hat{k}$$

$\vec{F}$ is conservative in $\mathbb{R}^2$ $\Longleftrightarrow$ potential fcn exists

$\Longleftrightarrow \frac{\partial}{\partial x}(y \text{ comp } \vec{F}) = \frac{\partial}{\partial y}(x \text{ comp } \vec{F})$

EX: $u(x,y) = x^2 - y^2 + xy + 3$ $\quad \Delta u \equiv 0$ ✓

Find v with $\nabla v = (-u_y, u_x) = (2y - x, 2x + y)$

Want v with
$$\begin{cases} \dfrac{\partial v}{\partial x} = 2y - x & (A) \\[2ex] \dfrac{\partial v}{\partial y} = 2x + y & (B) \end{cases}$$

Use A:

$$v = \int 2y - x \, dx = (2y)x - \tfrac{1}{2}x^2 + \underbrace{C(y)}$$

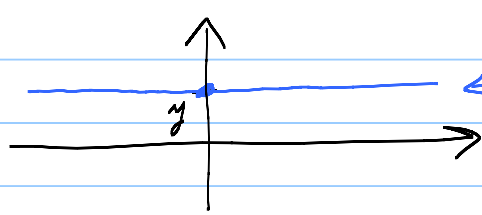
↑ "partial integral" w.r.t x

↑ arbitrary fcn of the other var(s)

Is this crazy? $\quad \dfrac{\partial}{\partial x}\left[v - \left[(2y)x - \tfrac{1}{2}x^2\right]\right]$

No!

$$= \frac{\partial v}{\partial x} - (2y - x^2) \equiv 0$$



$\leftarrow \frac{\partial}{\partial x} \equiv 0 \Rightarrow$ fcn const along horiz line

call it $C(y)$

We got from A: $\boxed{v = 2xy - \tfrac{1}{2}x^2 + C(y)}$

Now use B: $\quad \dfrac{\partial v}{\partial y} \overset{\text{want}}{=} 2x + y$

$$\frac{\partial}{\partial y}\Big(\underbrace{2xy-\tfrac{1}{2}x^2+c(y)}_{v \text{ from } A}\Big) \underset{\substack{\uparrow\\ \text{want}}}{=} 2x+y$$

$$2x - 0 + c'(y) = 2x+y$$

Miracle! all x's cancel: Get $\boxed{c'(y)=y}$

So $c(y)=\frac{1}{2}y^2+K$

Done! $\underline{v = 2xy-\frac{1}{2}x^2+\left(\frac{1}{2}y^2+K\right)}$ ← Harm conj is uniq up to addition of an arb const.

<u>Remark</u>: This method works on discs and $\mathbb{C}$.

Run into problem if there are holes in domain.

<u>EX</u>: $\ln(x^2+y^2)$ on