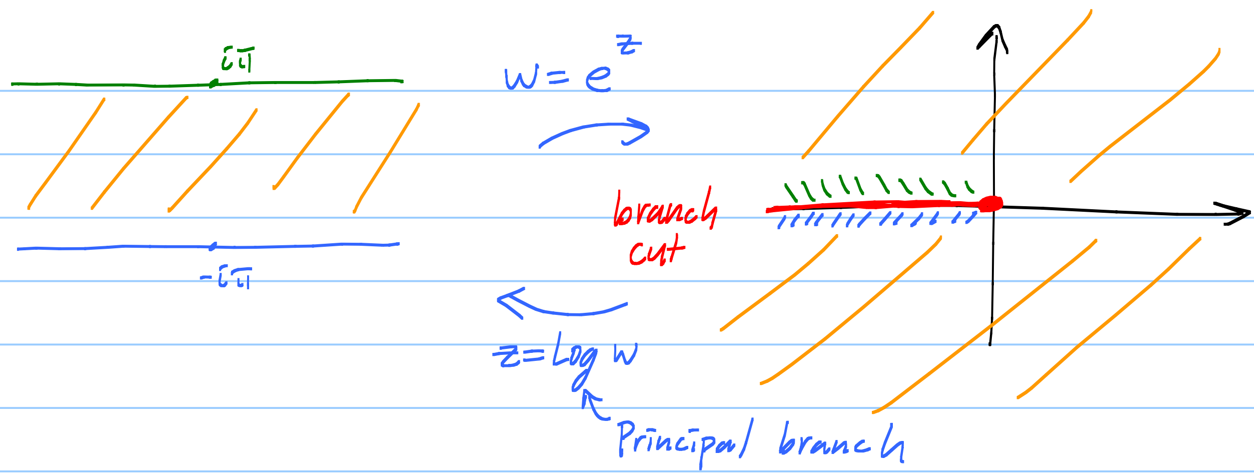
Lesson 10: Inverses of analytic functions



$z = \text{Log}\, w$: $\quad w = e^z = e^{x+iy} = e^x e^{iy} = re^{i\theta}$

$x+iy$

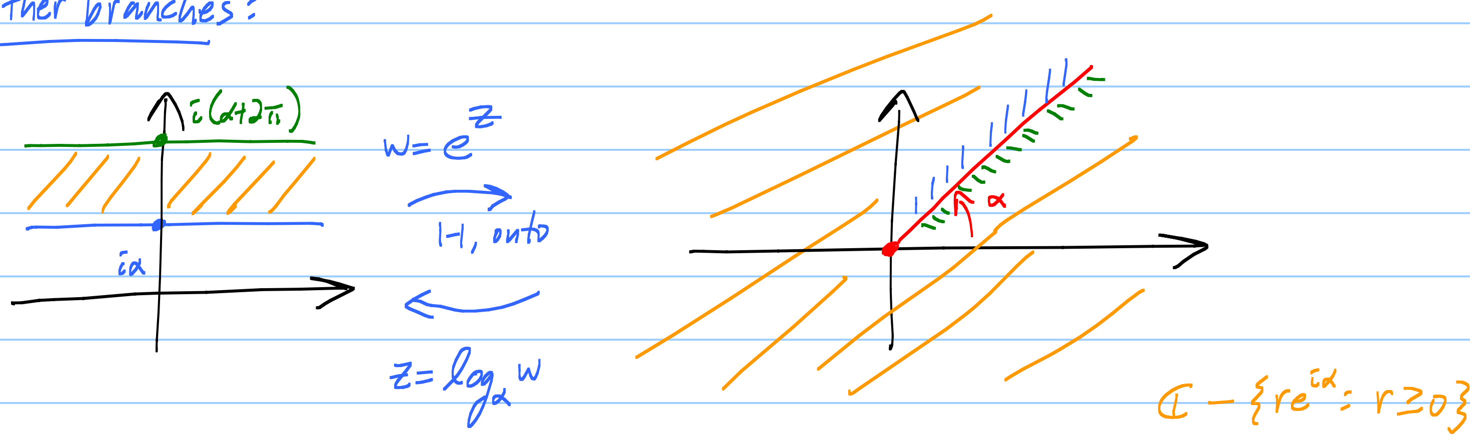
$|w| = e^x \longleftarrow x = \text{Ln}|w|$

$y \in \arg w \qquad y = \text{Arg}\, w \longleftarrow$ get Princ. Branch

Principal argument

Princ Branch : $\text{Log}\, w = \text{Ln}|w| + i\,\text{Arg}\, w$

Other branches:



Popular branch: $\log_0 w = \text{Ln}|w| + i\theta$ where $\theta \in \arg w$, $0 < \theta < 2\pi$

Fact: If f analytic and $f'(a) \neq 0$, then inverse of f is analytic near $f(a)$.

Why: $f(x+iy) = u(x,y) + i\,v(x,y)$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} u \\ v \end{pmatrix}$$

Jacobian matrix: $J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$

C-R Eqns

$\mathbb{R}^2 \to \mathbb{R}^2$ Inverse fcn thm:

Jacobian of inverse = inverse of Jacobian

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Aha! Preserves C-R Eqns!

Know if u, v are C^1-smooth, then corresponding U, V for inverse are C^1-smooth. u, v satisfy C-R Eqns $\Rightarrow$ so do U, V.

$\frac{d}{dz}(e^z) = e^z$ ← never zero   So $\log_\alpha$ is analytic.

Note: $e^{\log_\alpha z} = z$ ← always true

Chain Rule: $\frac{d}{dz}\left[e^{\log_\alpha z} \right] = \frac{d}{dz}[z] = 1$

$$\underbrace{E'(\log_\alpha z)}_{\substack{E(\log_\alpha z) \\ = z}} \frac{d}{dz}(\log_\alpha z) = 1$$

So $\boxed{\frac{d}{dz} \log_\alpha z = \frac{1}{z}}$

Warning: $\log e^z = z$ or $z + 2\pi n i$ $n \in \mathbb{Z}$ for some n.

Be careful: $\text{Log}\, z_1 z_2 = \text{Log}\, z_1 + \text{Log}\, z_2 + 2\pi n i$

for some $n \in \mathbb{Z}$ maybe.

Polar Cauchy-Riemann Equations:

$$f(x+iy) = u(x,y) + i\,v(x,y)$$
$$f(re^{i\theta}) = U(r,\theta) + i\,V(r,\theta)$$

$$U(r,\theta) = u(r\cos\theta,\ r\sin\theta)$$
$$V(r,\theta) = v(r\cos\theta,\ r\sin\theta)$$

Chain Rule: $U_r = u_x \cos x + u_y \sin\theta$

$$U_\theta = u_x(-r\sin\theta) + u_y(r\cos\theta)$$

$$\frac{1}{r}U_\theta = -u_x\sin\theta + u_y\cos\theta$$

Similarly for V.

$$\begin{bmatrix} U_r & V_r \\ \frac{1}{r}U_\theta & \frac{1}{r}V_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}$$

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \quad \begin{matrix} \Longleftarrow \\ \Longrightarrow \\ \text{too!} \end{matrix} \quad \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

Polar form of C-R Eqns: $$\boxed{\begin{aligned} U_r &= \frac{1}{r}V_\theta \\ V_r &= -\frac{1}{r}U_\theta \end{aligned}}$$

Ex: $\operatorname{Log} r^{i\theta} = \underbrace{\ln r}_{U} + i\underbrace{\theta}_{V}$

My notation: $\text{Ln}\, x$ = Real natural log

$\text{Log}\, z$ = Princ Branch

$\log z = \{w : e^w = z\}$

$\log z$ = a branch of complex log. (or $\log_\alpha z$)

Fantastic Fact: Trig functions and their inverses can all be expressed in terms of e^z and its inverse!

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Def^n:
$$\boxed{\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}}$$

$$\tan z = \frac{\sin z}{\cos z} \qquad \sinh z = \frac{e^z - e^{-z}}{2}$$

$$\tanh z = \frac{\sinh z}{\cosh z} \qquad \cosh z = \frac{e^z + e^{-z}}{2}$$

$$\cos^{-1} w = ?$$

Hmmm. $z = \cos^{-1} w$: $w = \cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

$$2w = e^{iz} + \frac{1}{e^{iz}} \quad \leftarrow \text{mult by } e^{iz}$$

$$2e^{iz} w = (e^{iz})^2 + 1$$

$$(e^{iz})^2 - (2w)e^{iz} + 1 = 0 \quad \leftarrow \text{Quadratic}$$

$$e^{iz} = \frac{2w + \sqrt{4w^2 - 4}}{2} = w + \sqrt{w^2 - 1}$$

$\leftarrow$ two complex sq. roots!

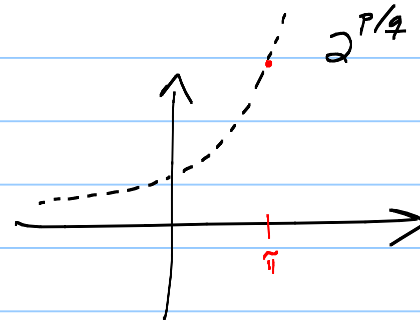
$$iz = \log\left(w + \sqrt{w^2 - 1}\right)$$

Aha! $z = \cos^{-1} w = \frac{1}{i} \log\left(w + \sqrt{w^2-1}\right)$

Similarly for all the other inverse fcns.

Liouville: Elementary fcns: e^z, $\log z$, algebraic functions and compositions.

Ouch! $\int e^{x^2}\,dx$ cannot be expressed as an elementary fcn!



Roots and powers for complex numbers:

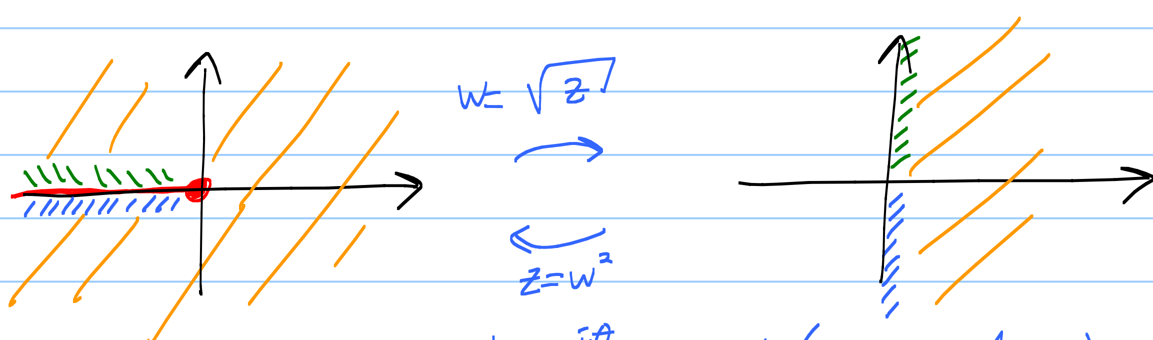
$$2^{\pi} = e^{\ln 2^{\pi}} = e^{\pi \ln 2}$$

$$z^p = e^{p \log z}$$

Principal branch: $z^p = e^{p \operatorname{Log} z}$

$\sqrt{z}$ = Principal square root $= e^{\frac{1}{2}\operatorname{Log} z}$

other square root = minus that



$$\sqrt{re^{i\theta}} = e^{\frac{1}{2}\operatorname{Log} re^{i\theta}} = e^{\frac{1}{2}(\ln r + i \operatorname{Arg} z)}$$

$$= e^{\frac{1}{2}\ln r}\, e^{i\frac{1}{2}\operatorname{Arg} z}$$

$$= \sqrt{r}\, e^{i\frac{1}{2}\operatorname{Arg} z}$$