Lesson 12 Complex integrals

Two kinds of fuctions: $z(t) : \mathbb{R} \to \mathbb{C}$ $\qquad z(t) = x(t) + iy(t)$

$f(z) : \mathbb{C} \to \mathbb{C}$ $\qquad f(x+iy) = u(x,y) + iv(x,y)$

Two kinds of derivatives: $z'(t) = x'(t) + iy'(t)$

$$f'(z) = \lim DQ = \begin{cases} u_x + iv_x \\ v_y - iu_y \end{cases} \quad \text{C-R Eqns}$$

Two kinds of integrals: $$\int_a^b z(t)\,dt = \int_a^b x(t)\,dt + i\int_a^b y(t)\,dt$$

$\gamma: z(t),\ a \le t \le b$ $$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,\underbrace{z'(t)\,dt}_{dz}$$

Two kinds of chain rules:

Chain rule #1: $h(z) = f(g(z))$ where f, g analytic, then
$$h'(z) = f'(g(z))\,g'(z)$$

Chain rule #2: f analytic and $z(t) = x(t) + iy(t)$,

then $$\frac{d}{dt} f(z(t)) = f'(z(t))\,z'(t)$$

Pf of #2: Could use same proof as I used for #1.

OR $$f(x(t) + iy(t)) = u(x(t), y(t)) + i\,v(x(t), y(t))$$

Now $$\frac{d}{dt} = [u_x \dot{x} + \underset{\overset{\|}{-v_x}}{u_y} \dot{y}] + i[v_x \dot{x} + \underset{\overset{\|}{u_x}}{v_y} \dot{y}] \quad \text{C-R Eqns}$$

$$= [u_x + iv_x] \cdot (\dot{x} + i\dot{y})$$

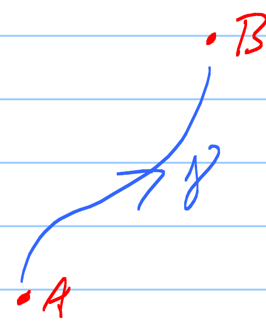
$$= f'(z(t))\; z'(t) \quad \checkmark$$

EX: $\dfrac{d}{dt} e^{i3t} = \dfrac{d}{dt} \underset{\uparrow\, E(z)=e^z}{E}(\underbrace{i3t}_{z(t)}) = \underbrace{E'(i3t)}_{E(i3t)} \underbrace{\dfrac{d}{dt}(i3t)}_{i3} = i3e^{i3t}$

Two Fund. Theorems of Calculus:

1) $\displaystyle\int_a^b z'(t)\,dt = z(b) - z(a)$

2) $\displaystyle\int_\gamma f'(z)\,dz = f(B) - f(A)$



Pf of #2: Assume f analytic on a domain containing γ.
[so u and v are nice and smooth.]

$$\int_\gamma f'(z)\,dz = \int_a^b \underbrace{f'(z(t))\; z'(t)}_{\text{Chain rule \#2!}}\,dt$$

$$\int_a^b \left[\frac{d}{dt} f(z(t))\right] dt \underset{\substack{\uparrow \\ \text{Fund Thm} \\ \text{Calc. \#1}}}{=} f(\underbrace{z(b)}_{B}) - f(\underbrace{z(a)}_{A}) \quad \checkmark$$

Two Basic Estimates:

1) $\displaystyle\left|\int_a^b z(t)\,dt\right| \le \int_a^b |z(t)|\,dt$

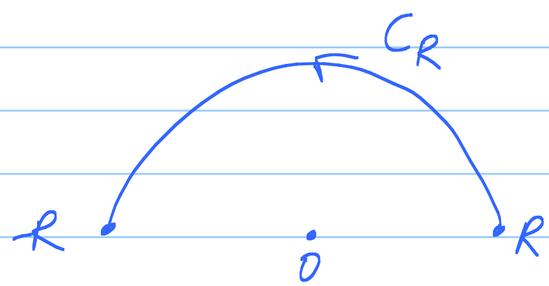
2) $\displaystyle\left|\int_\gamma f(z)\,dz\right| \le \underbrace{\left(\max_{z\in tr(\gamma)} |f(z)|\right)}_{M} \text{Length}(\gamma)$

$tr(\gamma) =$ "trace of γ" $= \{ z(t) : a \le t \le b \}$

Pf of 2: $\left| \int_\gamma f\, dz \right| = \left| \int_a^b f(z(t))\, z'(t)\, dt \right|$

$$\underset{\substack{\uparrow \\ \text{Basic Est.} \\ \#1}}{\le} \int_a^b \underbrace{\left| f(z(t)) \right|}_{\le M} \left| z'(t) \right| dt \le M \underbrace{\int_a^b |z'(t)|\, dt}_{\substack{\int_a^b \sqrt{\dot{x}^2 + \dot{y}^2}\, dt \\ \| \\ \text{Length}(\gamma)\,!}}$$

EX:



$C_R : z(t) = Re^{it}, \; 0 \le t \le \pi$

Prob: $\displaystyle\int_{C_R} z^2\, dz = \int_0^\pi \left[Re^{it} \right]^2 \underbrace{iRe^{it}\, dt}_{z'(t)\, dt}$

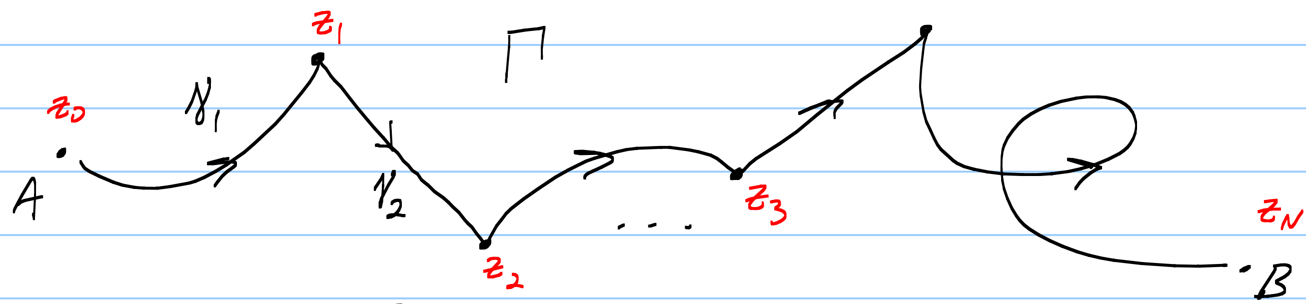
$$= \int_0^\pi R^2 e^{i2t}\, iRe^{it}\, dt = iR^3 \int_0^\pi e^{i3t}\, dt$$

$$= iR^3 \int_0^\pi \cos 3t + i \sin 3t \; dt = \cdots$$

Better way: $\displaystyle\int_{C_R} z^2\, dz = \int_{C_R} \frac{d}{dz}\left[\frac{1}{3} z^3 \right] dz$

$$= \frac{1}{3} z^3 \Big]_{\text{START}}^{\text{END}} = \frac{1}{3}(-R^3) - \frac{1}{3} R^3 = -\frac{2}{3} R^3$$

<u>Important fact</u>: Fund. Thm of Calc #2 holds for curves with corners [piecewise C^1-smooth curves].



$$\int_\Gamma f'\,dz = \sum \int_{\gamma_k} f'\,dz$$

$$= [\cancel{f(z_1)} - f(z_0)] + [\cancel{f(z_2)} - \cancel{f(z_1)}] + [\cancel{f(z_3)} - \cancel{f(z_2)}] + \cdots$$

$$+ [f(z_N) - \cancel{f(z_{N-1})}]$$

$$= f(z_N) - f(z_0) \quad \checkmark$$

Coming soon: Cauchy's Theorem. Our way: 4.4 b

<u>Big Fact</u>: $f(z)$ has an <u>analytic antiderivative</u> on a domain Ω if and only if $\int_\gamma f\,dz = 0$ for all closed curves in Ω.

<u>Why</u>: ($\Longrightarrow$) easy direction. Suppose $f(z) = F'(z)$ on Ω.

If γ is closed: $\int_\gamma f\,dz = \int_\gamma F'(z)\,dz \underset{\text{FTC \#2}}{=} F(\text{END}) - F(\text{START})$

$\underbrace{\text{END}\quad\text{START}}_{=}$

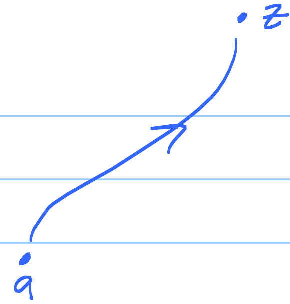
$= 0$

($\Longleftarrow$) Suppose f analytic and $\int_\gamma f\,dz = 0$ for all closed γ.

Hmmm. If I had F with $F' = f$, then

$$\int_{\gamma_A^B} f\,dz = \int_{\gamma_A^B} F'\,dz = F(B) - \underbrace{F(A)}_{\text{constant if start held fixed}}$$

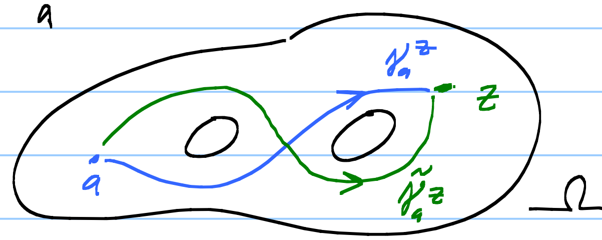
$B = z$



Big idea: Define

$$F(z) = \int_{\gamma_a^z} f(w)\,dw$$

Step 1: F is well defined!



$\Gamma = \gamma_a^z \cup (-\tilde{\gamma}_a^z)$ is a closed curve

$$0 = \int_\Gamma f\,dw = \int_{\gamma_a^z} f\,dw + \underbrace{\left(\int_{-\tilde{\gamma}_a^z} f\,dw\right)}_{-\int_{\tilde{\gamma}_a^z} f\,dw} = F(z) - \tilde{F}(z) \checkmark$$

Step 2: Use Basic Estimate #2 to show that

$$\frac{d}{dz}\int_{\gamma_a^z} f(w)\,dw = f(z) \leftarrow \text{Fund. Calc. \#3 !}$$