Lesson 14 The Cauchy Integral Formula

12, 13, 14 due Wed
Exam 1 in class Mon., Oct. 2
Lessons 1-16

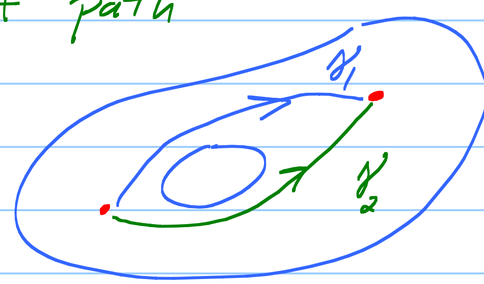
Theorem: Suppose f is analytic on a domain Ω. The following conditions are equivalent:

1) f has an analytic antiderivative on Ω given by
$$F(z) = \int_{\gamma_a^z} f(w)\,dw,$$

2) $\int_\gamma f\,dz = 0$ for any closed curve γ in Ω,

3) $\int_\Gamma f\,dz$ is independent of path

$1 \Leftrightarrow 2$ last time. $2 \Rightarrow 3$ easy:

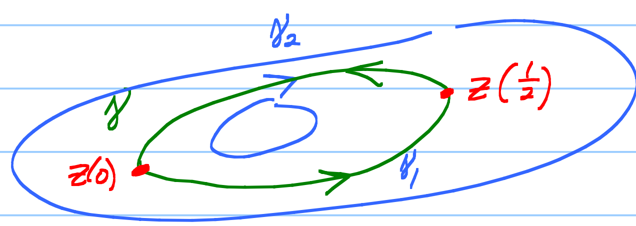


$\Gamma = \gamma_1 \cup (-\gamma_2)$ is a closed contour.

So $0 = \int_\Gamma f\,dz = \left(\int_{\gamma_1} + \int_{-\gamma_2}\right) f\,dz$. But $\int_{-\gamma_2} f\,dz = -\int_{\gamma_2} f\,dz$

So $\int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz$

$3 \Rightarrow 2$ easier.



$\gamma: z(t),\ 0 \le t \le 1$

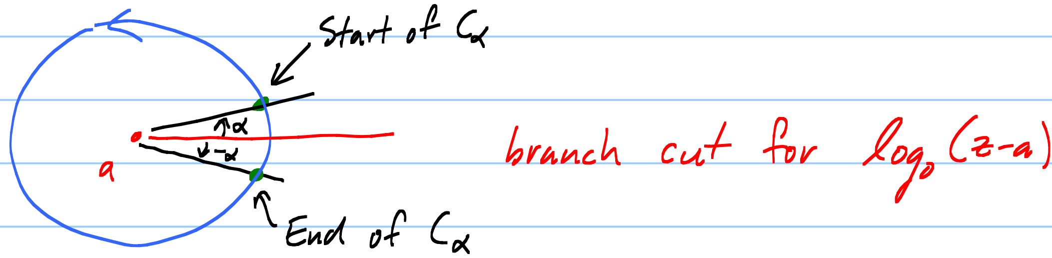
$\gamma_1: z(t),\ 0 \le t \le \frac{1}{2}$

γ_2: minus the curve $z(t),\ \frac{1}{2} \le t \le 1$

Calculation: Circle $C_R(a)$ centered at a: $z(t) = a + Re^{it},\ 0 \le t \le 2\pi$

Then $$\int_{C_R(a)} \frac{1}{z-a}\,dz = 2\pi i$$

Why:
$$\int_0^{2\pi} \frac{1}{(a+Re^{it})-a} \underbrace{iRe^{it}}_{z'(t)}\,dt = \int_0^{2\pi} i\,dt \quad \checkmark$$



$$\int_{C_R(a)} \frac{1}{z-a}\,dz = \lim_{\alpha\to 0} \int_{C_\alpha} \frac{1}{z-a}\,dz$$

$$= \lim_{\alpha\to 0} \int_{C_\alpha} \frac{d}{dz}\left[\log_0(z-a)\right] dz$$

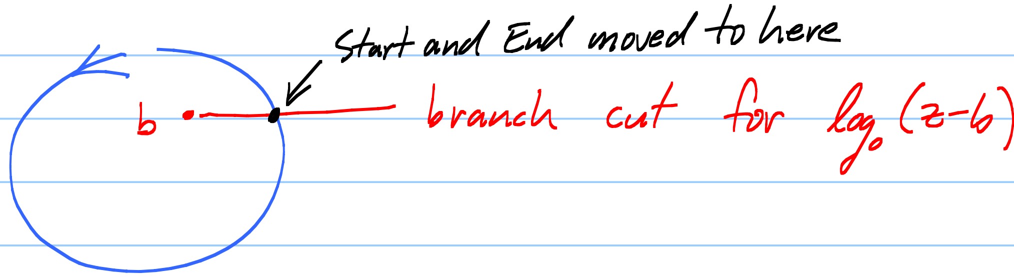
$$= \lim_{\alpha\to 0} \log_0(z-a) \Big|_{a+Re^{i\alpha}}^{a+Re^{i(2\pi-\alpha)}}$$

$$= \lim_{\alpha\to 0} \left\{ \left[\operatorname{Ln} R + i(2\pi-\alpha)\right] - \left[\operatorname{Ln} R + i\alpha\right] \right\}$$

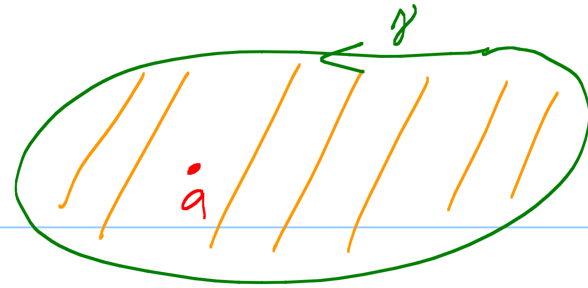
$$= \lim_{\alpha\to 0} i(2\pi - 2\alpha) = 2\pi i$$

Fact:
$$\int_{C_R(a)} \frac{1}{z-b}\,dz = \begin{cases} 2\pi i & \text{if } b \text{ inside circle} \\ 0 & \text{if } b \text{ outside by Cauchy's Thm.} \end{cases}$$

Why:



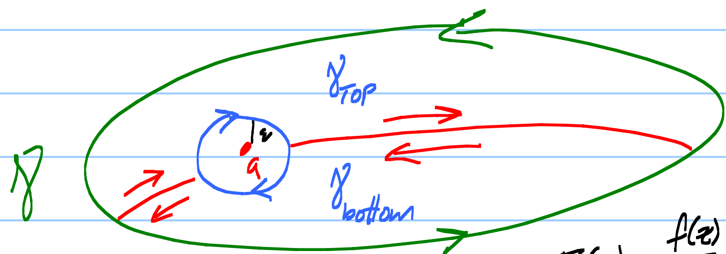
Cauchy Integral formula:



$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz$$

Assume γ is a CCW Jordan curve, f is analytic inside and on γ.

Pf:



$F(z) = \frac{f(z)}{z-a}$

Cauchy Thm:

$\int_{\gamma_{top}} F(z)\,dz = 0$, bottom too.

Aha! Add $\int_{\gamma_{top}}$ and $\int_{\gamma_{bottom}}$. Red integrals cancel!

Get $\int_\gamma F\,dz + \int_{-C_\varepsilon(a)} F\,dz = 0$

$$\underbrace{\int_{C_\varepsilon(a)} \frac{f(z)}{z-a}\,dz}_{\text{Claim } \to\, 2\pi i\, f(a) \text{ as } \varepsilon\to 0} = \int_\gamma \frac{f(z)}{z-a}\,dz$$

$$= \int_0^{2\pi} \frac{f(a+\varepsilon e^{it})}{(a+\varepsilon e^{it})-a}\, i\varepsilon e^{it}\,dt$$

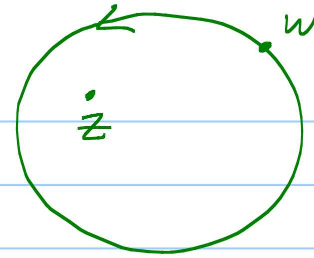
$$= i\int_0^{2\pi} \underbrace{f(a+\varepsilon e^{it})}_{\to f(a) \text{ as } \varepsilon\to 0.}\,dt$$

f analytic at a $\Rightarrow$ f cont. at a

$\longrightarrow i\, 2\pi\, f(a)$ ✓

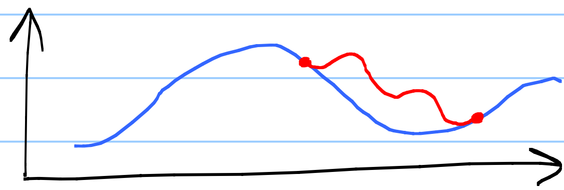
Cauchy Integral Formula for a disc:

$$f(z) = \frac{1}{2\pi i} \int_{C_R(a)} \frac{f(w)}{w-z}\, dw$$



Whoa! Value inside circle is determined by values on circle!

Way unlike $\mathbb{R} \to \mathbb{R}$:



Fantastic Fact: Can differentiate under $\int$ sign in Cauchy Int. Form! So f analytic $\Rightarrow f'$ analytic $\Rightarrow f''$...

Why: $$DQ = \frac{f(z)-f(a)}{z-a} = \frac{1}{2\pi i} \cdot \frac{1}{z-a} \int_\gamma \frac{f(w)}{w-z} - \frac{f(w)}{w-a}\, dw$$

$$= \frac{1}{2\pi i} \frac{1}{z-a} \int_\gamma f(w) \frac{\overbrace{(w-a)-(w-z)}^{z-a}}{(w-z)(w-a)}\, dw$$

pull out of $\int$ and cancel!

$$= \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)(w-a)}\, dw \xrightarrow{\text{expect}} \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^2}\, dz$$

as $z \to a$

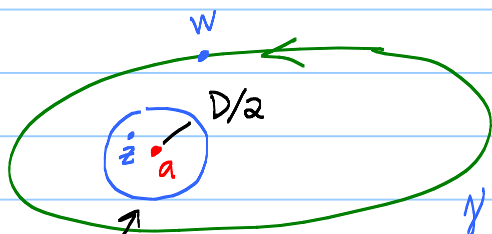
Check: $$\int_\gamma f(w) \left[\frac{1}{(w-z)(w-a)} - \frac{1}{(w-a)^2} \right] dw$$

$$= \int_\gamma f(w) \frac{z-a}{(w-a)^2(w-z)}\, dw$$

$$= (z-a) \underbrace{\int_\gamma \frac{f(w)}{(w-a)^2 (w-z)}\, dw}_{\text{need to see this stays bounded. Done!}}$$

$(z-a)$ goes to zero as $z \to a$

Basic Estimate: $\left| \int_\gamma \right| \le \left(\operatorname{Max}_\gamma |f| \right) \left[\dfrac{1}{D^2 \cdot \left(\frac{D}{2}\right)} \right] \operatorname{length}(\gamma)$



$D = \operatorname{dist}(a, \gamma) = \min_{\alpha \le t \le \beta} |z(t) - a|$

$\gamma:\ z(t),\ \alpha \le t \le \beta$

Then $\operatorname{dist}(z, \gamma) > \dfrac{D}{2}$

$\operatorname{Max}_\gamma |f| = \max \{ |f(z(t))| : \alpha \le t \le \beta \}$

continuous real fcn on $[\alpha, \beta]$ assumes its max.