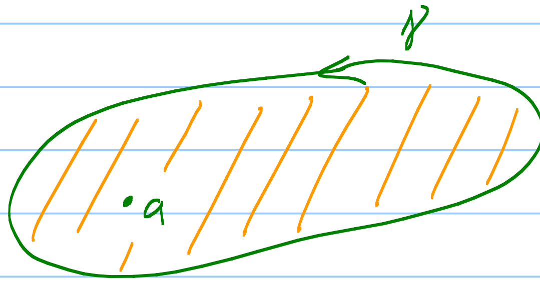
Lesson 15 Liouville's Theorem and the Fund. Thm. of Algebra

Exam 1 in class next Monday. Find practice problems on 425 Home Page.

12, 13, 14 due Wed.



$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz$$

$$f'(a) = \text{Lim } DQ = \text{Diff under integral} = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^2}\,dz$$

Can repeat!

$$f''(a) = \frac{2}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^3}\,dz$$

$$\vdots$$

Higher order Cauchy Theorem: $$\boxed{f^{(n)}(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz}$$

Fact: Suppose $\varphi : tr(\gamma) \to \mathbb{C}$ is merely continuous on γ.

Then $G(z) = \int_\gamma \frac{\varphi(w)}{w-z}\,dw$ is analytic on Ω and

$$G'(z) = \int_\gamma \frac{\varphi(w)}{(w-z)^2}\,dw.$$ [Same pf.]

Morera's Theorem: Suppose $f : \Omega \to \mathbb{C}$ is continuous and

$\int_\gamma f\,dz = 0$ for every closed curve γ in Ω.

Then f is analytic!

Why: $F(z) = \int_{\gamma_a^z} f(w)\,dw$ is well defined.

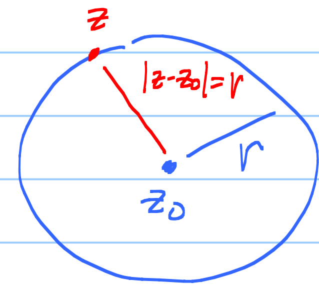
Same pf when f analytic shows that $F' = f$.

Aha! F is analytic. Cauchy Integral showed F' is analytic too!

Cauchy Estimates:

$$\left| f^{(n)}(z_0) \right| = \left| \frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right|$$

$$\leq \frac{n!}{2\pi} \left(\max_{C_r(z_0)} |f| \cdot \frac{1}{r^{n+1}} \right) \underbrace{\text{Length}(C_r(z_0))}_{2\pi r}$$



$$\boxed{\left| f^{(n)}(z_0) \right| \leq \frac{n!\, M}{r^n}}$$ where $M = \max_{C_R(z_0)} |f|$.

Liouville's Theorem: A bounded entire function must be constant.

Why: Assume f entire: analytic on $\mathbb{C}$

and bounded on $\mathbb{C}$: $\exists\, M > 0$ such that $|f(z)| \leq M$ on $\mathbb{C}$.

Step 1: Pick any old $z_0 \in \mathbb{C}$.

Step 2: Cauchy Estimate on $C_R(z_0)$:

$$\left| f'(z_0) \right| \leq \frac{1!\left(\max |f| \text{ on } C_R(z_0)\right)}{R^1} \leq \frac{M}{R}$$

Aha! Let $R \to \infty$. See $f'(z_0) = 0$.

But z_0 was arbitrary. So $f' \equiv 0$ on $\mathbb{C}$.

So $f \equiv$ const.

Way false $\mathbb{R} \to \mathbb{R}$. Ex: $\sin(t)$.

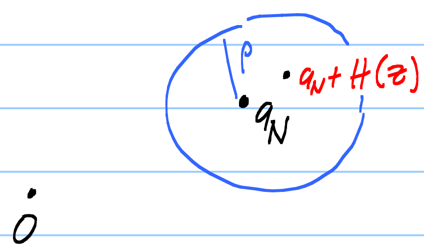
Basic Polynomial Estimate: $P(z) = a_N z^N + \cdots + a_1 z + a_0$ where $N \geq 1$. There are real constants $0 < A < B$ and a radius $R_0 > 0$ such that

$$A|z|^N \leq |P(z)| \leq B|z|^N \quad \text{when } |z| > R_0.$$

Why:

$$P(z) = z^N \Big[\underset{\neq 0}{a_N} + \underbrace{a_{N-1} \cdot \frac{1}{z} + \cdots + a_1 \cdot \frac{1}{z^{N-1}} + a_0 \cdot \frac{1}{z^N}}_{H(z)} \Big]$$

Note that $H(z) \to 0$ as $z \to \infty$.



$A = |a_N| - \rho$

$B = |a_N| + \rho$

when $|H(z)| < \rho$.

Hmmm; Why not take $\rho = \frac{|a_N|}{2}$.

[Note: taking ρ small makes A and B close to $|a_N|$!]

R_0 such that $|H(z)| < \rho$ if $|z| > R_0$.

$A = \frac{|a_N|}{2}$, $B = \frac{3|a_N|}{2}$.

Fund. Thm. Algebra: $P(z)$ $(N \geq 1)$ has a root in $\mathbb{C}$!

Can factor polys!

Why: Suppose $P(z)$ $(N \geq 1)$ has no zero in $\mathbb{C}$.

Then $\frac{1}{P(z)}$ is entire. Basic Poly estimate shows that $\frac{1}{P(z)} \to 0$ as $|z| \to \infty$. This shows that

$\frac{1}{p(z)}$ is a bounded entire fcn! Liouville's $\Longrightarrow$

$\frac{1}{p(z)} \equiv$ const. So $p(z) = $ const. Crazy!

Because $\frac{d^N}{dz^N} P(z) = N!\, a_N \neq 0$. Conclude $P(z)$

must have a zero.

Note: Pf that $\frac{1}{p(z)}$ is bound on $\mathbb{C}$.

$$\left|\frac{1}{p(z)}\right| \leq \frac{1}{A|z|^N} \quad \text{for } |z| > R_0$$

So $\left|\frac{1}{p(z)}\right| \to 0$ as $|z| \to \infty$. So $\exists\, R > 0$

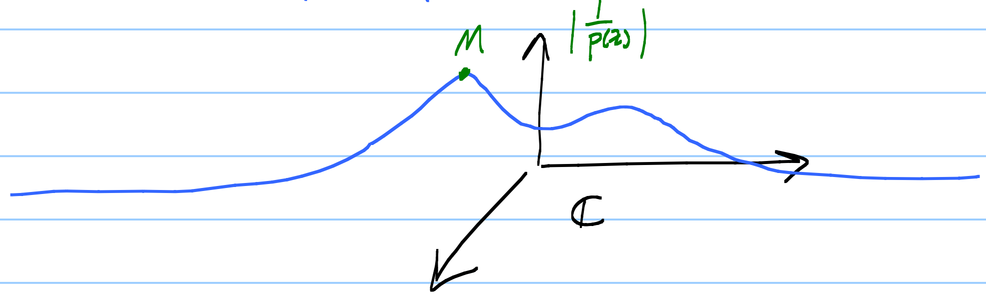
such that $\left|\frac{1}{p(z)}\right| < 1$ if $|z| > R$.

Aha! But $\left|\frac{1}{p(z)}\right|$ is a continuous function on

the closed and bounded set $\overline{D_R(0)}$ (compact set),

so it assumes its max M at a pt in the set.

Now $\left|\frac{1}{p(z)}\right| \leq \operatorname{Max}(M, 1)$.



Fact: Analytic functions satisfy the averaging property:

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\, dt$$

<u>Why</u>: $$f(z_0) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(z)}{z - z_0}\, dz \qquad C_r(z_0):\ z(t) = z_0 + re^{it}, \quad z'(t) = ire^{it}$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{[\underbrace{z_0 + re^{it}}_{z(t)}] - z_0} \underbrace{(ire^{it})}_{z'(t)}\, dt$$

$$= \checkmark$$

Consequently Harmonic fcns do too :

u harmonic. Get conjugate v. $f = u + iv$.

Write ave int. Separate into real and imag parts.