Lesson 16 Max Principle

Exam 1 in class Monday.
Review on Friday

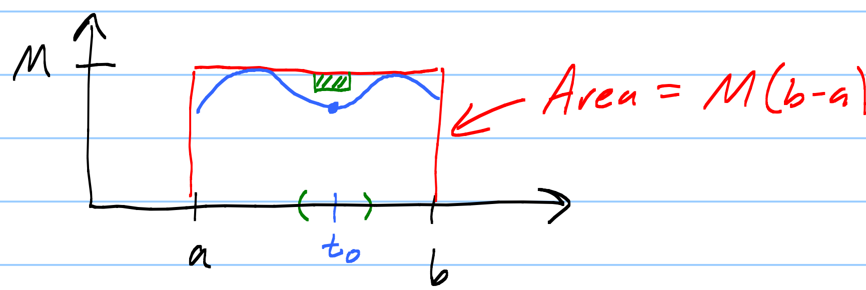
Cauchy Integral $\Rightarrow$ Ave Prop : $f(z_0) = \underbrace{\frac{1}{2\pi r}}_{\text{length}} \int_0^{2\pi} f(z_0 + re^{it}) \underbrace{r\,dt}_{ds}$

Fact: Ave Prop for harmonic fcns follows:
u harmonic. Get harm conj v. Ave Prop for $f = u + iv$ holds. Take real part. Get ave prop for u.

Ave Prop $\Rightarrow$ Max Princ.

Lemma 1: Calculus Lemma. Suppose $0 \le h(t) \le M$ on $[a,b]$, and suppose $h : [a,b] \to \mathbb{R}$ is continuous. Then, if
$$\int_a^b h \, dt = M(b-a), \quad \text{then} \quad h(t) \equiv M \text{ on } [a,b].$$

Why:



Lemma: If f is analytic on $D_R(a)$ and $|f|$ is constant there, then $f \equiv$ const too.

Why: $f = u + iv$. $\quad |f|^2 = u^2 + v^2 = c$. $\quad (*)$

Case $c = 0$: Then $f \equiv 0$. ✓

Case $c \neq 0$: $\frac{\partial}{\partial x}(*)$: $\quad 2u\,u_x + 2v\,v_x = 0$

$$\frac{\partial}{\partial y}(*) : \qquad 2u\,u_y + 2v\,v_y = 0$$

C-R Eqns

$$\underbrace{\begin{bmatrix} u_x & v_x \\ u_y^{\;-v_x} & v_y^{\;u_x} \end{bmatrix}}_{\text{det must } = 0!} \begin{pmatrix} u \\ v \end{pmatrix} = \vec{0}$$

not the zero vector $(u^2+v^2=c)$

Aha! $\quad \text{Det} = (u_x)^2 + (v_x)^2 \equiv 0$

So $\quad u_x \equiv 0, \quad v_x \equiv 0$

$u_x = v_y \equiv 0 \qquad u_y = -v_x \equiv 0$

Get $\quad \nabla u \equiv 0$

$\nabla v \equiv 0$

So $f = u + iv$ is const.

Theorem: (Max Princ).

I) Suppose f is analytic on a domain Ω and suppose that $|f|$ assumes its max at a point z_0 inside Ω. Then $f \equiv$ const.

[No mountain tops in Ω on $|f|$ graph]

II) Furthermore, if Ω is a bounded domain and f is continuous up to the boundary, then the max of $|f|$ must happen on the boundary.

$$\left(|f(z)| \text{ on } \Omega\right) \leq \underset{\text{Boundary of } \Omega}{\text{Max}} |f|.$$

Why: I) Suppose $|f|$ max at $z_0 \in \Omega$.

$\exists R > 0$ with $D_R(z_0) \subset \Omega$.

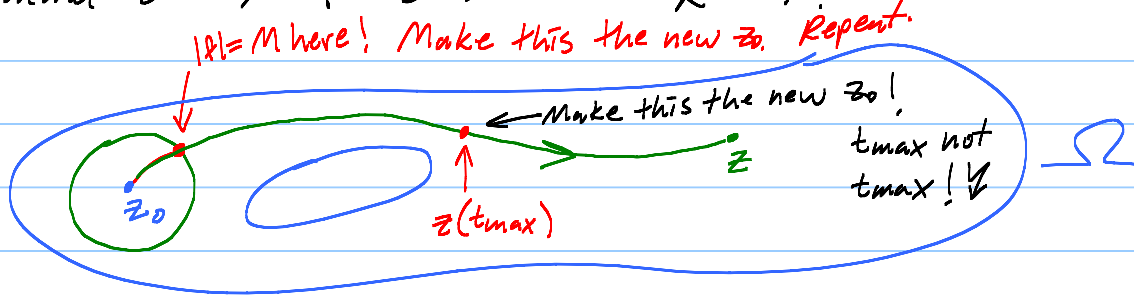
Ave Prop at z_0: $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\,dt$

$$2\pi \underbrace{|f(z_0)|}_{M} \leq \int_0^{2\pi} \underbrace{|f(z_0+re^{it})|}_{\leq M}\,dt$$

Calc. Lemma $\Rightarrow$ $|f(z_0+re^{it})| \equiv M$ for $0<r<R$.

So $|f| = \text{const}$ on $D_R(z_0)$.

Lemma 2 $\Rightarrow$ f const on $D_R(z_0)$!



$\gamma: z(t): \alpha \leq t \leq \beta$.

Let t_{max} be the supremum of $\tilde{\tau}$ such that

$|f(z(t))| = M$ for all t with $\alpha \leq t < \tilde{\tau}$.

Note $t_{max} > \alpha$ from above. Claim: $t_{max} = \beta$.

Why: If $t_{max} < \beta$: Continuity $\Rightarrow$ $|f(z(t_{max}))| = M$

t_{max} not t_{max}. ↯. So $t_{max} = \beta$. ✓

So $|f| \equiv M$ on Ω. Lemma $\Rightarrow$ $\nabla u \equiv 0$, $\nabla v \equiv 0$ on Ω.

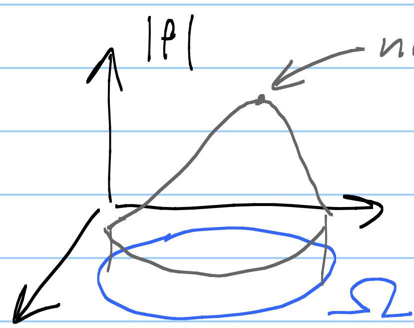
$\Rightarrow u, v$ const on Ω. ✓

I) $\Rightarrow$ II). If f is continuous, then $|f|$ continuous,

and a continuous fcn on a <u>closed</u>, <u>bounded</u> $\overline{\Omega}$

assumes its max. If max happens at a point inside Ω, then I $\Rightarrow$ f const. So max occurs on boundary too. If max happens at boundary pt., then ✓.

<u>Picture:</u>



<u>Note:</u> Analytic function can have zeroes inside Ω.

Looks like no Min Princ. <u>But</u>

<u>Theorem:</u> If f has no zeroes in Ω, then the Minimum princ holds.

<u>Why:</u> Because then $\frac{1}{f}$ is analytic and a max of $\left|\frac{1}{f}\right|$ happens at a min of $|f|$.

<u>Max and Min Principal holds for harmonic fcns.</u>

Repeat same proof. $(\text{Max of } -u) = (\text{Min of } u)$.

<u>or</u> u harmonic. Get harmonic conjugate v.

$f = u + iv$ analytic. $F = e^{u+iv}$ is analytic.

$$|F| = \underbrace{|e^u|}_{e^u}\ \underbrace{|e^{iv}|}_{=1} = e^u$$

Max of e^u occurs at max of u!

Liouville's Theorem for harmonic fcns: Suppose u is harmonic on $\mathbb{C}$ and $u \leq M$ on $\mathbb{C}$. Then u is constant on $\mathbb{C}$.

Why: Cook up harm conj v on $\mathbb{C}$. $f = u + iv$ entire.

$$F = e^{u+iv} \quad \text{entire}$$

$$|F| = e^{u} \leq e^{M}. \quad \text{Liouville's} \Rightarrow F \text{ const.}$$

So $|F| = e^{u} \equiv$ const. So $u = \ln(\text{const}) =$ Const.

Dirichlet Problem: Given a temp fcn $\varphi(e^{it})$ on the unit circle, find steady state temp inside.

i.e., find a harmonic function u that is continuous up to the unit circle such that $u(e^{it}) = \varphi(e^{it})$.

Fact: Max, Min Princ $\Rightarrow$ sol^n u is unique.

Why: Suppose u_1 and u_2 both solve prob. Then $u_1 - u_2$ is harmonic on $D_1(0)$, continuous on $\overline{D_1(0)}$, and $\equiv 0$ on $C_1(0)$. But Max and Min of $u_1 - u_2$ happen on $C_1(0)$! So $u_1 - u_2 \begin{smallmatrix} \leq 0 \\ \geq 0 \end{smallmatrix}$ on $\overline{D_1(0)}$.

So $u_1 - u_2 \equiv 0$ and $u_1 \equiv u_2$. ✓

Cauchy Integral Formula for harmonic fcns :

Poisson Formula :

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|a|^2}{|a-e^{it}|^2} \, u(e^{it}) \, dt$$