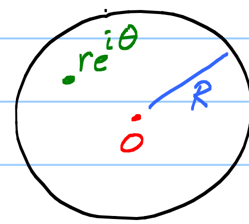
Lesson 17 Review

Exam 1 in class Monday
Closed book, no notes, calculators, phones, watches, ...

Poisson Formula: u harmonic on $D_\rho(0)$, $\rho > 1$, then

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|a|^2}{|e^{i\theta}-a|^2}\, u(e^{i\theta})\, d\theta$$

Other forms: On disc $D_R(0)$:



$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)}{R^2+r^2-2rR\cos(t-\theta)}\, u(Re^{it})\, dt$$

Fantastic Fact: Poisson Formula solves the Dirichlet Problem!

Big facts to know:

1) Ω domain, f analytic on Ω:

Equiv:
A) $\int_\gamma f\, dz = 0$ for closed γ in Ω

B) $f = F'$ for analytic F on Ω

C) $\int_{\gamma_a^b} f\, dz$ is indep of path.

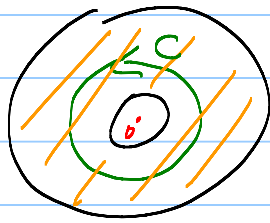
2) Cauchy Theorem: a) If Ω is simply connected domain and f analytic on Ω, then

$$\int_\gamma f\, dz = 0 \quad \text{for any closed } \gamma \text{ in } \Omega.$$

b) f analytic inside and on a Jordan curve γ, then $\int_\gamma f\, dz = 0$.

3) If u is harmonic on a simply connected domain Ω, then u has a harmonic conj on Ω.

Hole:



$\ln|z| = \operatorname{Re} \log z$ ← branch cut!

↑ no harmonic conj.

$$\int_C \frac{1}{z}\,dz = 2\pi i$$

Prob 4: $f: \mathbb{C} \to \mathbb{R}$ analytic. Must be constant.

$$f(x+iy) = u(x,y) + i\underbrace{v(x,y)}$$

↑ "Real valued" means $v \equiv 0$.

C-R Eqns: $\begin{cases} u_x = v_y \equiv 0 \\ u_y = -v_x \equiv 0 \end{cases}$ So $\nabla u \equiv 0$

So $u \equiv$ const. $v \equiv 0$. So $f \equiv$ const.

Works for f on a domain Ω.

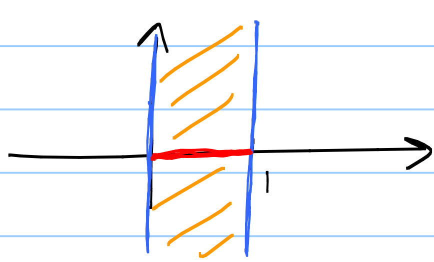
f analytic $\Rightarrow$ f continuous $\Rightarrow$ u, v continuous

and f analytic $\Rightarrow$ u_x, u_y, v_x, v_y exist

and satisfy C-R Eqns.

Thm from class. u continuous, u_x, u_y exist and $\nabla u \equiv 0$ on a domain $\Rightarrow$ u const.

Prob 6d:



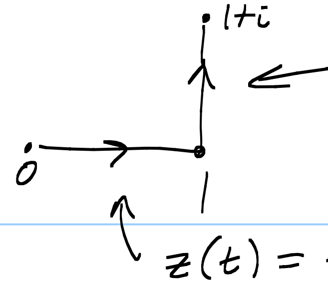
Open, yes.

Connected, no.

Not a domain

Prob. 10 :

a)



$z(t) = 1 + it \qquad 0 \le t \le 1$

$z'(t) = i$

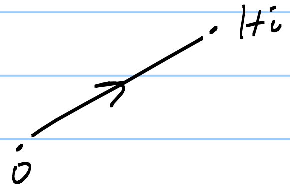
$z(t) = t \qquad 0 \le t \le 1$

$z'(t) = 1$

$|zw| = |z| \cdot |w|$

$$\int_\gamma |z|^2 dz = \int_0^1 |z(t)|^2 \, z'(t) \, dt + \int_0^1 \text{second} \ldots$$

b)

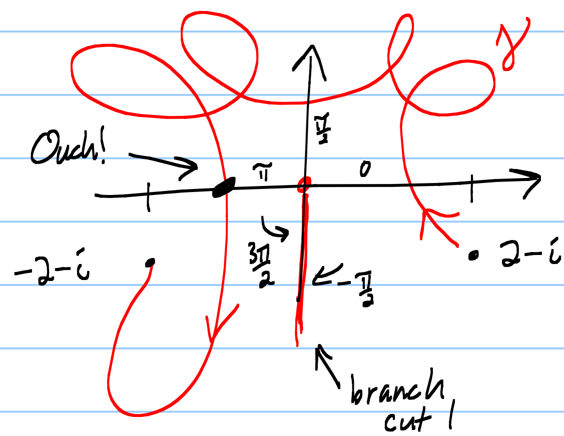


$z(t) = \underset{\substack{\uparrow \\ A=0}}{A} + (\underset{\substack{\uparrow \\ B=1+i}}{B} - A)t \quad : 0 \le t \le 1$

$z'(t) = B - A = 1+i$

Integrals $\ne$! OK, because $|z|^2$ not analytic.

Prob. 11 :



$\frac{1}{z^3} = F'(z)$

where $F(z) = -\frac{1}{2} \frac{1}{z^2}$

$$\int_\gamma \frac{1}{z^3} dz = F(-2-i) - F(2-i)$$

Can't use Princ. Branch $\operatorname{Log} z$ to compute $\int_\gamma \frac{1}{z} dz$.

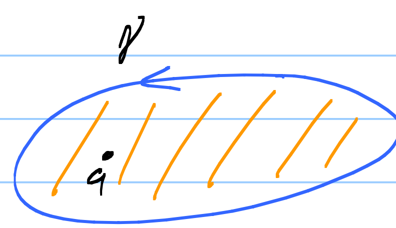
Use branch : $\log z = \log_{-\frac{\pi}{2}} z = \ln|z| + i\theta$

$\theta \in \arg z$

$-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$

Then $\frac{d}{dz}(\log z) = \frac{1}{z}$ on γ.

Cauchy Integral Formula problems :



$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz$$

Probs: $\displaystyle\int_{C_r(0)} \frac{\sin iz}{z^2-4}\, dz \qquad 0 < r < 2.$ Zero by Cauchy.

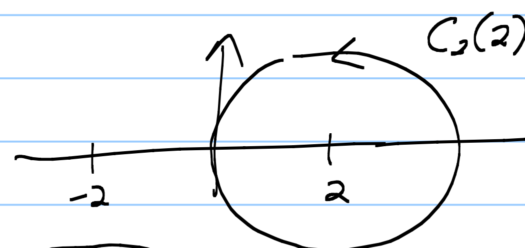
What if $r > 2$? $\displaystyle\frac{1}{z^2-4} = \frac{1}{(z-2)(z+2)} = \frac{A}{z-2} + \frac{B}{z+2}$

Reduce to C.I.F. $\quad f(z) = (\text{Const.}) \sin iz$

$$\sin iz = \frac{e^{i(iz)} - e^{-i(iz)}}{2i} = \frac{e^{-z} - e^{z}}{2i} = -\frac{1}{i} \sinh z$$

$$\frac{d}{dz}(\sin iz) = \cos(iz) \cdot \frac{d}{dz}(iz) = i \cos iz$$

What if $C = C_2(2)$?



$$\int_{C_2(2)} \frac{\sin iz}{z^2-4}\, dz = \int_{C_2(2)} \frac{\sin iz}{(z-2)(z+2)}\, dz = 2\pi i\, f(2)$$

$f(z)$!

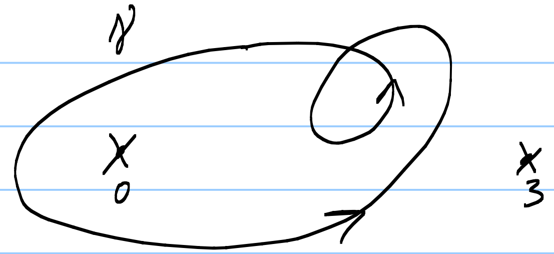
What if: $\displaystyle\int_{C_{10}(0)} \frac{\sin iz}{(z-2)^3}\, dz \qquad 3 = n+1 \quad \boxed{n=2}$

$f(z) = \sin iz$

$$f^{(2)}(a) = \frac{2!}{2\pi i} \int_{C_{10}(0)} \frac{f(z)}{(z-a)^3}\, dz$$

Ans: $\frac{2\pi i}{2!} f^{(2)}(2)$ where $f(z) = \sin iz$.

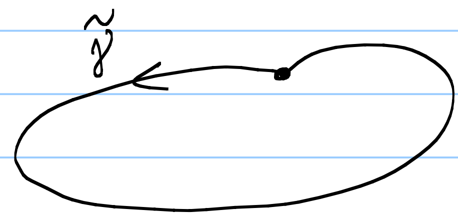
HWK:



$$\int_\gamma \frac{\sin iz}{z(z-3)}\,dz$$

$\int = 0$ about the little closed loop.

Toss it out.



Cauchy Int. Form. $f(z) = \frac{\sin iz}{z-3}$

$a = 0$

$z = (z-0)$!

14.